On the ℓ_2-induced control for eventually periodic systems

Mazen Farhood and Geir E. Dullerud

***Abstract*— This paper is focused on the ℓ_2-induced control of eventually periodic linear discrete-time systems. We prove for such systems that a synthesis exists if and only if an eventually periodic synthesis exists. We also consider specific cases, where we show that the synthesis if existent can always be chosen to be of the same eventually periodic class as the plant. All the conditions derived are provided in terms of semi-definite programming problems. The motivation for this work is controlling nonlinear systems along prespecified trajectories, notably those which eventually settle down into periodic orbits and those with uncertain initial states.**

I. Introduction

In this paper, we continue our work started in [3] on the control of *eventually periodic* systems. Such systems are aperiodic for an initial amount of time, and then become periodic afterwards. Our work is motivated by the desire to use robust control methods for the control of nonlinear systems along prespecified trajectories. There are two basic ways in which eventually periodic dynamics arise when linearizing systems along trajectories: (1) the system trajectory is an aperiodic maneuver joined to a subsequent periodic orbit; or (2) the initial condition of the system is uncertain. We remark that both finite horizon and periodic systems are subclasses of eventually periodic systems.

One of the main contributions of [3] is the derivation of necessary and sufficient conditions for the existence of eventually periodic controllers, exhibiting equal transient time variation and periodicity as the plant; such controllers both stabilize and provide performance in closed-loop control systems. However, the invalidity of the said synthesis conditions does not necessarily imply the non-existence of a synthesis. In this paper, we improve on this result and show that, for an eventually periodic plant, the existence of a synthesis is equivalent to the existence of an eventually periodic synthesis, having the same periodicity as the plant but probably exhibiting longer transient time variation. Furthermore, we consider certain cases where the synthesis if existent can always be chosen to be of the same eventually periodic class as the plant.

The general machinery used to obtain the results of this paper is motivated by the work in [4], [8], [9], combined with the time-varying system machinery developed in [2]. Also, see the closely related earlier work in [1], [6], [7] on nonstationary systems. The literature in the area of time-varying systems is vast, and we refer the reader to [5] for a comprehensive list of general references.

This work was supported by NSF ECS-98-75244

The authors are with the Department of Mechanical and Industrial Engineering, University of Illinois, Urbana, IL 61801, USA

II. Preliminaries

We now introduce our notation and gather some elementary facts. The set of real numbers and that of real $n\times m$ matrices are denoted by $\mathbb{R}$ and $\mathbb{R}^{n\times m}$ respectively. If S_i is a sequence of operators, then $\mathrm{diag}(S_i)$ denotes their block-diagonal augmentation.

Given two Hilbert spaces E and F, we denote the space of bounded linear operators mapping E to F by $\mathfrak{L}(E,F)$, and shorten this to $\mathfrak{L}(E)$ when E equals F. If X is in $\mathfrak{L}(E,F)$, we denote the E to F induced norm of X by $\|X\|_{E\to F}$; when the spaces involved are obvious, we write simply $\|X\|$. The adjoint of X is written X^*. When an operator $X\in\mathfrak{L}(E)$ is self-adjoint, we use $X<0$ to mean it is negative definite; that is there exists a number $\alpha>0$ such that, for all nonzero $x\in E$, the inequality $\langle x,Xx\rangle<-\alpha\|x\|^2$ holds.

We will be primarily concerned with two Hilbert spaces in this paper. The first is the standard space $\mathbb{R}^n$ with the inner product given by $\langle x,y\rangle_{\mathbb{R}^n}=\sum_{t=0}^{n-1}x_ty_t=x^*y$. The second Hilbert space of interest is formed given an infinite sequence $\{\mathbb{R}^{n_t}\}$ of Hilbert spaces, and is denoted by $\ell_2(\{\mathbb{R}^{n_t}\})$. It is defined as the subspace of the Hilbert space direct sum $\oplus_{t=0}^{\infty}\mathbb{R}^{n_t}$ consisting of elements $(x_0,x_1,x_2,\dots)$ which satisfy $\sum_{t=0}^{\infty}\|x_t\|^2_{\mathbb{R}^{n_t}}<\infty$. The inner product of x and y in $\ell_2(\{\mathbb{R}^{n_t}\})$ is defined by the infinite sum $\langle x,y\rangle_{\ell_2}=\sum_{t=0}^{\infty}\langle x_t,y_t\rangle_{\mathbb{R}^{n_t}}$. In the sequel, we will frequently suppress the subscript on the dimension symbol n_t and accordingly use a shorter notation for $\ell_2(\{\mathbb{R}^{n_t}\})$, namely $\ell_2(\mathbb{R}^n)$. Also, when the spatial dimensions n_t are either evident or not relevant to the discussion, we abbreviate further to ℓ_2. We will use $\|x\|$ to denote $\sqrt{\langle x,x\rangle}$, the standard norm on this space.

One of the most important operators used in the paper is the unilateral shift operator Z defined as follows:

$$\begin{array}{rcll} Z: \ \ell_2(\{\mathbb{R}^{m_k}\}) & \to & \ell_2(\{\mathbb{R}^{n_k}\}), & \text{where } \ m_k=n_{k+1} \\ (a_0,a_1,a_2,\dots) & \overset{Z}{\longmapsto} & (0,a_0,a_1,a_2,\dots). & \end{array}$$

Following the notation and approach in [2], we start by making the following definitions.

Definition 1: *A bounded linear operator Q mapping $\ell_2(\{\mathbb{R}^{m_k}\})$ to $\ell_2(\{\mathbb{R}^{n_k}\})$ is* block-diagonal *if there exists a sequence of matrices Q_k in $\mathbb{R}^{n_k\times m_k}$ such that, for all w,z, if $z=Qw$, then $z_k=Q_kw_k$. Then Q has the representation* $\mathrm{diag}(Q_0,Q_1,Q_2,\dots)$.

Definition 2: *An operator P on ℓ_2 is (h,q)-eventually periodic if, for some non-negative integer h, we have*

$$Z^q((Z^*)^hPZ^h)=((Z^*)^hPZ^h)Z^q. \qquad (1)$$

In the case where $q=1$, P is called h-eventually time-invariant. Also, when only the period length q is relevant, we simply call P eventually q-periodic.

Note that when $h = 0$, equality (1) reduces to the following: $Z^q P = P Z^q$. Hence, in such a case, P simply commutes with the q-shift, and we accordingly refer to P as a q-periodic operator. Throughout the sequel we set $h \geq 0$ and $q \geq 1$ to be some *fixed* integers.

We denote the *first period truncation* of a q-periodic *block-diagonal* operator Q by $\hat{Q}$, and define such a matrix as $\hat{Q} := \mathrm{diag}(Q_0, \ldots, Q_{q-1})$. Also, we define the cyclic shift matrix $\hat{Z}$ for $q \geq 2$ by

$$\hat{Z} = \begin{bmatrix} 0 & \cdots & 0 & I \\ I & \ddots & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix},$$

so that $\hat{Z}^* \hat{Q} \hat{Z} = \mathrm{diag}(Q_1, \ldots, Q_{q-1}, Q_0)$. For $q = 1$, set $\hat{Z} = I$.

Now suppose Q is an (h,q)-eventually periodic block-diagonal operator, then we define the matrix $\tilde{Q}$ to be the *finite-horizon-first-period truncation* of Q, namely $\tilde{Q} := \mathrm{diag}(Q_0, \ldots, Q_{h-1}, Q_h, \ldots, Q_{h+q-1})$. Furthermore, in the sequel, we will use the notation $\Xi(\tilde{Q})$ to denote the matrix $\mathrm{diag}(Q_1, \ldots, Q_{h+q-1}, Q_h)$.

Having established these definitions, we are now ready to consider the main subject of this paper.

III. Eventual periodicity of the synthesis

Let G be a linear time-varying discrete-time system defined by the following state space equation:

$$\begin{bmatrix} x_{k+1} \\ z_k \\ y_k \end{bmatrix} = \begin{bmatrix} A_k & B_{1k} & B_{2k} \\ C_{1k} & D_{11k} & D_{12k} \\ C_{2k} & D_{21k} & 0 \end{bmatrix} \begin{bmatrix} x_k \\ w_k \\ u_k \end{bmatrix} \quad x_0 = 0, \qquad (2)$$

for $w \in \ell_2$. The signals x_k, z_k, w_k, y_k, and u_k are real and have time-varying dimensions which we denote by n_k, n_{zk}, n_{wk}, n_{yk}, and n_{uk} respectively. For notational simplicity, in the following we frequently suppress the time-dependence of the above dimensions. We make the assumption that all the state space matrices are uniformly bounded functions of time, and further assume the direct feedthrough term $D_{22} = 0$. Also, we assume that the block-diagonal operators, defined by the sequences of the above state space matrices, are (h,q)-eventually periodic.

We suppose this system is being controlled by a controller K whose state space equation is

$$\begin{bmatrix} x^K_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} A^K_k & B^K_k \\ C^K_k & D^K_k \end{bmatrix} \begin{bmatrix} x^K_k \\ y_k \end{bmatrix} \quad x^K_0 = 0.$$

The controller state vector $x^K_k \in \mathbb{R}^r$ where the time dependence of r is suppressed. Again, we assume that the block-diagonal operators, defined by the matrix sequences A^K_k, B^K_k, C^K_k, and D^K_k, are (h,q)-eventually periodic. The connection of G and K is shown in Figure 1. Since $D_{22} = 0$, this interconnection is always well-posed.

We write the realization of the closed-loop system as

$$\begin{aligned} x^L_{k+1} &= A^L_k x_k + B^L_k w_k \\ z_k &= C^L_k x_k + D^L_k w_k, \end{aligned} \qquad (3)$$

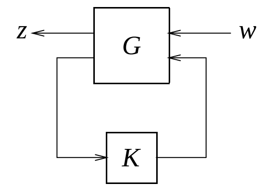

Fig. 1. Closed-loop system

where x^L_k contains the combined states of G and K, and A^L_k, B^L_k, C^L_k and D^L_k are appropriately defined. Here $A^L_k \in \mathbb{R}^{(n+r)\times(n+r)}$, where n is the number of states of G and r is the number of states of K. Note that the block-diagonal operators A^L, B^L, C^L and D^L are (h,q)-eventually periodic. The above closed-loop system may be written more compactly in operator form as

$$\begin{aligned} x^L &= Z A^L x^L + Z B^L w \\ z &= C^L x^L + D^L w, \end{aligned} \qquad (4)$$

where Z is the shift operator on ℓ_2. Assuming the relevant inverse exists, we can write the map from w to z as

$$w \mapsto z = C^L (I - Z A^L)^{-1} Z B^L + D^L.$$

It is possible to show that $I - Z A^L$ has a bounded inverse if and only if the system $x^L_{k+1} = A^L_k x^L_k$ is exponentially stable.

Remark 3: *All finite horizon systems are stable due to the underlying assumption that the state space matrices are uniformly bounded functions of time. Hence, as far as stability is concerned, an (h,q)-eventually periodic system can be regarded as a q-periodic system with an initial condition that is basically the final state, x_h, obtained from the finite horizon part. Thus, the stability of an eventually periodic system boils down to the stability of its periodic part, which, in turn, is equivalent to the invertibility of the matrix $I - \hat{Z}\hat{A}^L_{per}$, where $\hat{A}^L_{per} = \mathrm{diag}(A^L_h, \ldots, A^L_{h+q-1})$.*

The following definition expresses our synthesis goal.

Definition 4: *A controller K is an* admissible synthesis *to G in Fig.1 if $I - Z A^L$ has a bounded inverse and the closed-loop performance inequality $\|w \mapsto z\|_{\ell_2 \to \ell_2} < 1$ is achieved.*

We now state the following result from [3], which gives finite dimensional convex conditions, the validity of which is equivalent to the existence of an admissible (h,q)-eventually periodic synthesis.

Theorem 5: *Suppose that G is (h,q)-eventually periodic. There exists an admissible (h,q)-eventually periodic synthesis K for G with state dimension $r \leq n$ if and only if there exist block-diagonal matrices $\tilde{R} > 0$ and $\tilde{S} > 0$ satisfying*

$$\begin{bmatrix} \tilde{V}_1^* & \tilde{V}_2^* \end{bmatrix} \left\{ E \begin{bmatrix} \tilde{R} & \\ & I \end{bmatrix} E^* - \begin{bmatrix} \Xi(\tilde{R}) & \\ & I \end{bmatrix} \right\} \begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} < 0, \qquad (5)$$

$$\begin{bmatrix} \tilde{U}_1^* & \tilde{U}_2^* \end{bmatrix} \left\{ E^* \begin{bmatrix} \Xi(\tilde{S}) & \\ & I \end{bmatrix} E - \begin{bmatrix} \tilde{S} & \\ & I \end{bmatrix} \right\} \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{bmatrix} < 0, \qquad (6)$$

$$\begin{bmatrix} \tilde{R} & I \\ I & \tilde{S} \end{bmatrix} \geq 0, \qquad (7)$$

where $E = \begin{bmatrix} \tilde{A} & \tilde{B}_1 \\ \tilde{C}_1 & \tilde{D}_{11} \end{bmatrix}$, and, for $i = 0, \ldots, h+q-1$,

$$\begin{aligned} \mathrm{Im} \begin{bmatrix} V_{1i}^* & V_{2i}^* \end{bmatrix}^* &= \mathrm{Ker} \begin{bmatrix} B_{2i}^* & D_{12i}^* \end{bmatrix}, \quad V_{1i}^* V_{1i} + V_{2i}^* V_{2i} = I, \\ \mathrm{Im} \begin{bmatrix} U_{1i}^* & U_{2i}^* \end{bmatrix}^* &= \mathrm{Ker} \begin{bmatrix} C_{2i} & D_{21i} \end{bmatrix}, \quad U_{1i}^* U_{1i} + U_{2i}^* U_{2i} = I. \end{aligned}$$

Solutions $\tilde{R}$ and $\tilde{S}$ can be used to construct an (h,q)-eventually periodic controller K. The way to construct this controller can be found in [2], [4], [9].

We remark that if the aforesaid synthesis conditions are invalid, we can only say that there exists no admissible *(h,q)-eventually periodic synthesis*; but this does not necessarily imply the non-existence of a different admissible synthesis. We will next show that the existence of an admissible synthesis for an (h,q)-eventually periodic plant is equivalent to the existence of an admissible (N,q)-eventually periodic synthesis, for some $N \geq h$. But first, we define the set $\mathfrak{X}$ to consist of positive definite operators X of the form

$$X = \mathrm{diag}(X_0, X_1, X_2, \ldots) > 0, \text{ where } X_i \in \mathbb{R}^{n_i \times n_i}. \quad (8)$$

Theorem 6: *Given an (h,q)-eventually periodic plant G, then there exists an admissible synthesis K for G if and only if there exists an admissible eventually q-periodic synthesis.*

Proof: The proof of the "if" direction is immediate. Following is the proof of the "only if" direction. From [2], we know that an admissible synthesis K exists for G, with state dimension $r \leq n$, if and only if there exist operators $R, S \in \mathfrak{X}$ satisfying the following synthesis conditions:

$$F^*RF - V_1^*Z^*RZV_1 + H < 0, \quad (9)$$

$$J^*Z^*SZJ - U_1^*SU_1 + W < 0, \quad (10)$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0, \quad (11)$$

where $F = A^*V_1 + C_1^*V_2$, $J = AU_1 + B_1U_2$, $H = M^*M - V_2^*V_2$, $M = B_1^*V_1 + D_{11}^*V_2$, $W = L^*L - U_2^*U_2$, $L = C_1U_1 + D_{11}U_2$, and U_i, V_i are defined as in Theorem 5.

Clearly, all of the operators in (9) and (10) are (h,q)-eventually periodic block-diagonal operators. Furthermore, from our definition of negative definite operators, there exists a sufficiently small $\beta > 0$ such that the left-hand sides of both (9) and (10) are each less than $-\beta I$. It turns out that only the last instance of the finite horizon will be relevant to this proof, and so, we will assume henceforth, without loss of generality, that the finite horizon length h is equal to 1. Then the state space operator A will have the representation:

$$A = \mathrm{diag}(A_0, A_{per}), \text{ with } A_{per} = \mathrm{diag}(\hat{A}_{per}, \hat{A}_{per}, \ldots), \quad (12)$$

$\hat{A}_{per}$ being the first period truncation of the q-periodic block-diagonal operator A_{per}. Similar representations apply for the other state space operators.

Now since there exists an admissible synthesis for the (h,q)-eventually periodic plant, then definitely there exists an admissible synthesis for the q-periodic portion of this plant. Then invoking Theorem 22 of [2], we deduce that there exists an admissible q-periodic synthesis for this periodic part, and consequently, there exists positive definite block-diagonal matrices $\hat{R}_{per}$ and $\hat{S}_{per}$ satisfying

$$\hat{F}_{per}^*\hat{R}_{per}\hat{F}_{per} - \hat{V}_{1,per}^*\hat{Z}^*\hat{R}_{per}\hat{Z}\hat{V}_{1,per} + \hat{H}_{per} < -\beta I,$$

$$\hat{J}_{per}^*\hat{Z}^*\hat{S}_{per}\hat{Z}\hat{J}_{per} - \hat{U}_{1,per}^*\hat{S}_{per}\hat{U}_{1,per} + \hat{W}_{per} < -\beta I,$$

$$\begin{bmatrix} \hat{R}_{per} & I \\ I & \hat{S}_{per} \end{bmatrix} \geq 0.$$

Then following a similar argument to that of the proof of Lemma 7 in [3], we can construct (N,q)-eventually periodic solutions to the synthesis conditions (9), (10) and (11), for some $N \geq h$, and use these solutions to form an admissible (N,q)-eventually periodic synthesis for the plant G. ■

The question at this point is whether we can find at least reasonable upper bounds on the finite horizon lengths for such eventually q-periodic syntheses. This is not an easy problem in the general case. However, if we are to consider specific systems, such as those with exactly measurable states or those in which the state disturbance is a linear transformation of the sensor noise at each instance of the finite horizon, then this problem becomes viably solvable. In fact, it turns out that the synthesis condition (9), which we will refer to hereafter as the forward synthesis condition (FSC), always admits an (h,q)-eventually periodic solution if feasible. This, however, cannot be said for the backward synthesis condition (10), which we simply abbreviate as BSC. Moreover, even if (9) and (10) both admit solutions in the subclass of the (N,q)-eventually periodic operators of $\mathfrak{X}$ for some $N \geq h$, none of these solutions might satisfy the coupling condition (11), and hence we may need to settle for a larger finite horizon length. However, in the case of the said specific systems, the BSC simplifies significantly and then the main focus becomes to show that the FSC admits an (h,q)-eventually periodic solution in $\mathfrak{X}$ if feasible.

We conclude this section with further comments on the BSC. As previously mentioned, given an (h,q)-eventually periodic plant, the existence of a solution in $\mathfrak{X}$ to the BSC does not necessarily imply the existence of an (h,q)-eventually periodic solution in $\mathfrak{X}$. However, if we are to drop the constraint that the solution has to be positive definite, and expand our search for solutions to the set $\mathfrak{X}_e$, whose elements are bounded, self-adjoint and of the same form as those of the set $\mathfrak{X}$, but without the positive definiteness restriction, then we have the following result.

Theorem 7: *Given an (h,q)-eventually periodic plant, there exists a solution in $\mathfrak{X}_e$ to the BSC if and only if there exists an (h,q)-eventually periodic solution in $\mathfrak{X}_e$.*

The outline of the proof is as follows: first, we apply a generalized version of Finsler's lemma to the BSC, and then we adopt a similar argument to that used in the proof of the KYP lemma given in [3].

IV. Forward Synthesis Condition

This section shows that, given an (h,q)-eventually periodic system G, then the feasibility of its FSC implies the existence on an (h,q)-eventually periodic solution in $\mathfrak{X}$ satisfying the said condition. To start, suppose that there exists an operator $R \in \mathfrak{X}$ solving the FSC, namely

$$F^*RF - V_1^*Z^*RZV_1 + H < 0,$$

where the above system operators are all (h,q)-eventually periodic and block-diagonal, as defined in the preceding proof. Then, it is obvious from Theorem 6 that there exists an (N,q)-eventually periodic solution in $\mathfrak{X}$ to the FSC, for some $N \geq h$. The next two subsections are devoted to proving that $N = h$.

A. Technical machinery

We now develop some new tools that are essential to proving the main result of this section. We start by defining the following sets for non-negative integers i:

$$\mathbb{D}_i = \{X > 0 : V_{1i}^* X V_{1i} - H_i > 0\},$$
$$\mathbb{P}^{n_i} = \{X \in \mathbb{R}^{n_i \times n_i} : X = X^* \geq 0\}.$$

Also, for integers $i \geq 0$, we define the sequence of maps $\Omega_i : \mathbb{D}_i \to \mathbb{P}^{n_i}$ by $\Omega_i(X) = F_i(V_{1i}^* X V_{1i} - H_i)^{-1} F_i^*$.

Proposition 8: *Suppose $X > 0$, $i \geq 0$, and $Y^{-1} \in \mathbb{D}_i$. If $X \leq Y$, then $X^{-1} \in \mathbb{D}_i$ and $\Omega_i(X^{-1}) \leq \Omega_i(Y^{-1})$.*

Proof: Given that $X, Y > 0$, then applying the Schur complement formula twice to the condition $X \leq Y$, we get the equivalent inequality: $X^{-1} \geq Y^{-1}$. Then, from the latter inequality, together with the fact that $Y^{-1} \in \mathbb{D}_i$, we have

$$V_{1i}^* X^{-1} V_{1i} - H_i \geq V_{1i}^* Y^{-1} V_{1i} - H_i > 0. \tag{13}$$

Hence, $X^{-1} \in \mathbb{D}_i$. Now, applying the Schur complement formula twice to (13), we get the following:

$$(V_{1i}^* X^{-1} V_{1i} - H_i)^{-1} \leq (V_{1i}^* Y^{-1} V_{1i} - H_i)^{-1}.$$

Pre- and post-multiplying both sides of this inequality by F_i and F_i^* respectively, we obtain the following sought result: $\Omega_i(X^{-1}) \leq \Omega_i(Y^{-1})$. ■

Proposition 9: *Suppose $R \in \mathfrak{X}$ solves the FSC. Then, for each $i \geq 0$, both $R_{i+1} \in \mathbb{D}_i$ and $\Omega_i(R_{i+1}) + \varepsilon I < R_i^{-1}$ for some sufficiently small $\varepsilon > 0$.*

Proof: By assumption $R \in \mathfrak{X}$ and satisfies the FSC. Then the following holds:

$$V_1^* Z^* R Z V_1 - H > F^* R F \geq 0,$$

Hence, $V_{1i}^* R_{i+1} V_{1i} - H_i > 0$ for all integers $i \geq 0$, and consequently, $R_{i+1} \in \mathbb{D}_i$. Applying the Schur complement formula to the FSC, we get

$$\begin{bmatrix} -(V_1^* Z^* R Z V_1 - H) & F^* \\ F & -R^{-1} \end{bmatrix} < 0,$$

which, by invoking the Schur complement formula again, leads to the inequality $F(V_1^* Z^* R Z V_1 - H)^{-1} F^* - R^{-1} < 0$. Then, from our definition of negative definite operators, we know that there exists a sufficiently small $\varepsilon > 0$ such that the left-hand side of the last inequality is less than $-\varepsilon I$. Thus, $\Omega_i(R_{i+1}) + \varepsilon I < R_i^{-1}$ for all integers $i \geq 0$. ■

Throughout the remainder of this subsection, we fix ε to be some positive real number. For $q \geq 2$, we define the domain set $\hat{\mathbb{D}}$ by

$$\hat{\mathbb{D}} = \{X \in \mathbb{D}_{h+q-1} : \Theta \in \mathbb{D}_{i-1} \text{ for all } i = h+1, \ldots, h+q-1\},$$

where $\Theta = \Omega_i\left(\left[\Omega_{i+1}\left(\left[\cdots\left[T_\theta\right]^{-1}\cdots\right]^{-1}\right) + \varepsilon I\right]^{-1}\right) + \varepsilon I$,

with $T_\theta = \Omega_{h+q-2}\left(\left[\Omega_{h+q-1}(X) + \varepsilon I\right]^{-1}\right) + \varepsilon I$.

For $q = 1$, we set $\hat{\mathbb{D}} = \mathbb{D}_{h+q-1}$. Associated with this domain is the map $\hat{\Omega} : \hat{\mathbb{D}} \to \mathbb{P}^{n_h}$ defined by

$$\hat{\Omega}(X) = \Omega_h\left(\left[\Omega_{h+1}\left(\left[\cdots\left[T_\theta\right]^{-1}\cdots\right]^{-1}\right) + \varepsilon I\right]^{-1}\right),$$

where T_θ is defined as before. Last, for some integer $m \geq 1$, we formally define $\hat{\Omega}^m(X)$ by

$$\hat{\Omega}^m(X) = \underbrace{\hat{\Omega}([\hat{\Omega}([\cdots[\hat{\Omega}([\hat{\Omega}}_{\text{m times}}(X) + \varepsilon I]^{-1}) + \varepsilon I]^{-1}\cdots]^{-1}) + \varepsilon I]^{-1}).$$

Pertaining to the map $\hat{\Omega}$, we have the following two very important facts that follow directly from Proposition 8.

Corollary 10: *Suppose $X > 0$ and $Y^{-1} \in \hat{\mathbb{D}}$.*

(i) *If $X \leq Y$, then $X^{-1} \in \hat{\mathbb{D}}$ and $\hat{\Omega}(X^{-1}) \leq \hat{\Omega}(Y^{-1})$;*

(ii) *If $\hat{\Omega}(Y^{-1}) \leq Y$, then, for all $m \geq 1$, the following is true: $\hat{\Omega}^{m+1}(Y^{-1}) \leq \hat{\Omega}^m(Y^{-1}) \leq Y$.*

Part (i) of the claim follows routinely by an iterative application of Proposition 8; Part (ii) is easily shown by applying Part (i). We accordingly omit the proof.

A very useful corollary of Proposition 8 and Proposition 9 follows; recall that h and q are fixed in this section.

Corollary 11: *Suppose $R \in \mathfrak{X}$ solves the FSC. Then, for all $m \geq 1$, $R_{h+mq} \in \hat{\mathbb{D}}$ and $\hat{\Omega}(R_{h+mq}) + \varepsilon I < R_{h+(m-1)q}^{-1}$ for some sufficiently small $\varepsilon > 0$.*

The proof is immediate and so is not included.

B. Main Result

Now we can state the main result of this section.

Theorem 12: *Given an (h, q)-eventually periodic plant, then a solution in $\mathfrak{X}$ exists for the FSC if and only if an (h, q)-eventually periodic solution in $\mathfrak{X}$ exists.*

Proof: The proof of the "if" direction is immediate. We now prove the "only if" direction. It turns out that only the last instance of the finite horizon will be relevant to this proof, and so, we may assume without loss of generality that the finite horizon length h is equal to 1. Then the state space operator A will have the following representation: $A = \text{diag}(A_0, \hat{A}_{per}, \hat{A}_{per}, \ldots)$, where $\hat{A}_{per}$ is defined as in (12). Similar representations apply for the other system operators.

Now, by assumption, the FSC has a solution in $\mathfrak{X}$. Then, by invoking Theorem 6, there exists an eventually q-periodic operator R satisfying the FSC such that, for some non-negative integer N,

$$R = \text{diag}(R_0, R_1, \ldots, R_{Nq}, \bar{R}, \bar{R}, \ldots),$$

where $\bar{R} = \text{diag}(R_{Nq+1}, \ldots, R_{(N+1)q})$. Invoking Proposition 9 and Corollary 11, we deduce that the FSC holds only if the following sequence of inequalities holds for some sufficiently small $\varepsilon > 0$:

$$\begin{gathered} \Omega_0(R_1) < R_0^{-1} \\ \hat{\Omega}(R_{q+1}) + \varepsilon I < R_1^{-1} \\ \vdots \\ \hat{\Omega}(R_{Nq+1}) + \varepsilon I < R_{(N-1)q+1}^{-1} \\ \hat{\Omega}(R_{Nq+1}) + \varepsilon I < R_{Nq+1}^{-1}. \end{gathered} \tag{14}$$

Starting with the second last inequality, we can successively apply part (i) of Corollary 10 to obtain the inequality

$$\hat{\Omega}^N(R_{Nq+1}) + \varepsilon I < R_1^{-1}.$$

Invoking Proposition 8, the preceding inequality, along with the inequality $\Omega_0(R_1) < R_0^{-1}$ from (14), guarantees the validity of the following:

$$\Omega_0\left(\left[\hat{\Omega}^N(R_{Nq+1}) + \varepsilon I\right]^{-1}\right) < R_0^{-1}.$$

Set $Q = \left[\hat{\Omega}^N(R_{Nq+1}) + \varepsilon I\right]^{-1}$, then $\Omega_0(Q) < R_0^{-1}$. Also, appealing to part (ii) of Corollary 10, we deduce that

$$\hat{\Omega}^{N+1}(R_{Nq+1}) = \hat{\Omega}\left(\left[\hat{\Omega}^N(R_{Nq+1}) + \varepsilon I\right]^{-1}\right) \le \hat{\Omega}^N(R_{Nq+1}).$$

Defining

$$\Gamma_i\left(\hat{\Omega}^N\left(R_{Nq+1}\right)\right) = \Omega_i\left(\left[\Omega_{i+1}\left(\left[\cdots\left[T_\gamma\right]^{-1}\cdots\right]^{-1}\right) + \varepsilon I\right]^{-1}\right),$$

where $T_\gamma = \Omega_{q-1}\left(\left[\Omega_q(Q) + \varepsilon I\right]^{-1}\right) + \varepsilon I$, for $i = 2,\ldots,q$, we can equivalently write the above inequality as

$$\Omega_1\left(\left[\Gamma_2\left(\hat{\Omega}^N\left(R_{Nq+1}\right)\right) + \varepsilon I\right]^{-1}\right) \le \hat{\Omega}^N\left(R_{Nq+1}\right) < Q^{-1}.$$

Also, we have $\Gamma_i\left(\hat{\Omega}^N\left(R_{Nq+1}\right)\right) < \Gamma_i\left(\hat{\Omega}^N\left(R_{Nq+1}\right)\right) + \varepsilon I$ for $i = 2,\ldots,q$, which leads to the following:

$$\Omega_i\left(\left[\Gamma_{i+1}\left(\hat{\Omega}^N\left(R_{Nq+1}\right)\right) + \varepsilon I\right]^{-1}\right) < \Gamma_i\left(\hat{\Omega}^N\left(R_{Nq+1}\right)\right) + \varepsilon I,$$

for $i = 2,\ldots,q-1$, and

$$\Omega_q(\underbrace{[\hat{\Omega}^N(R_{Nq+1}) + \varepsilon I]^{-1}}_{Q}) < \Gamma_q\left(\hat{\Omega}^N\left(R_{Nq+1}\right)\right) + \varepsilon I.$$

Therefore, the $(1,q)$-eventually periodic operator $R_{eper} = \text{diag}(R_0, \hat{R}_{per}, \hat{R}_{per}, \ldots) \in \mathfrak{X}$ solves the FSC, where $\hat{R}_{per} = \text{diag}(Q, (\Gamma_2(\hat{\Omega}^N(R_{Nq+1})) + \varepsilon I)^{-1}, \ldots, (\Gamma_q(\hat{\Omega}^N(R_{Nq+1})) + \varepsilon I)^{-1})$. Thus, we have shown that, given an (h,q)-eventually periodic plant, we can always construct from any solution of the FSC an (h,q)-eventually periodic solution. ■

V. Special cases

As mentioned earlier, given an (h,q)-eventually periodic plant G, the feasibility of the BSC does not in general imply the existence of an (h,q)-eventually periodic operator in $\mathfrak{X}$ that solves the said condition; it does imply, however, that an eventually q-periodic solution in $\mathfrak{X}$ exists. While finding uppers bounds on the finite horizon lengths of such eventually q-periodic solutions is one of the main goals of this work, we will restrict our attention in the current paper to situations where the BSC simplifies significantly. This section consists of two subsections, where the first considers systems with exactly measurable states (i.e. $C_2 = I, D_{21} = 0$), and the second deals with plants in which the state disturbance is a linear transformation of the sensor noise at each instance of the finite horizon.

A. Exactly measurable states

Suppose G is an (h,q)-eventually periodic plant with exactly measurable states. Then, $C_2 = I, D_{21} = 0$, and the BSC simplifies to the following: $B_1^* Z^* SZB_1 < I - D_{11}^* D_{11}$. Now since, in this case, $D^L = D_{11} + D_{12}D^K D_{21} = D_{11}$, then in order for an admissible synthesis to exist, it is necessary that we have $\|D_{11}\| < 1$. Hence, in such a scenario, the right-hand side of the above linear operator inequality is always positive definite, and so we can always find a solution $S \in \mathfrak{X}$ satisfying the BSC. However, the choice of this solution can not really be arbitrary due to the coupling condition (11).

Lemma 13: *The following are equivalent:*

(i) *There exist operators $S, R \in \mathfrak{X}$ such that*

$$B_1^* Z^* SZB_1 < I - D_{11}^* D_{11}, \tag{15}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \ge 0; \tag{16}$$

(ii) *There exists $R \in \mathfrak{X}$ such that*

$$B_1 (I - D_{11}^* D_{11})^{-1} B_1^* < Z^* RZ. \tag{17}$$

Proof: We first prove that (i) implies (ii). Applying the Schur complement formula to (16), we get $Z^* SZ \ge Z^* R^{-1} Z$, which, together with (15), leads to the inequality $B_1^* Z^* R^{-1} ZB_1 < I - D_{11}^* D_{11}$. Applying the Schur complement formula to the preceding inequality gives (17). To prove that (ii) implies (i), we simply set $R = S^{-1}$, and then a couple of applications of the the Schur complement formula lead to (15) and (16). ■

Theorem 14: *Suppose that plant G is (h,q)-eventually periodic with $C_2 = I$ and $D_{21} = 0$. Then the existence of an admissible synthesis for G is equivalent to the existence of an (h,q)-eventually periodic* static *synthesis, which in turn is equivalent to the existence of a block-diagonal matrix $\tilde{R} > 0$ satisfying*

$$\tilde{F}^* \tilde{R} \tilde{F} - \tilde{V}_1^* \Xi(\tilde{R}) \tilde{V}_1 + \tilde{H} < 0,$$
$$\tilde{B}_1 (I - \tilde{D}_{11}^* \tilde{D}_{11})^{-1} \tilde{B}_1^* < \Xi(\tilde{R}).$$

Proof: The proof of the "if" direction is immediate. We now prove the "only if" direction. Suppose there exists a synthesis K for G. Then, by Lemma 13, the synthesis conditions (9), (10), and (11) are equivalent to the FSC and (17), which, by Theorem 6, admit an (N,q)-eventually periodic solution $R \in \mathfrak{X}$ for some $N \ge h$. Appealing to the proof of Theorem 12, we can construct an (h,q)-eventually periodic operator $R_{eper} \in \mathfrak{X}$ that solves the FSC and, due to the way we form this operator, also satisfies inequality (17). Now, since implicitly we have $R_{eper} = S_{eper}^{-1}$, and from [2], [4], [9] we know that we can construct a controller with state dimension $r_i = \text{rank}(I - R_{eper,i} S_{eper,i})$, then clearly we can form a static (h,q)-eventually periodic controller. ■

Once we find a solution $\tilde{R}$ to the synthesis conditions of Theorem 14, we can solve the following linear matrix inequality for $\tilde{D}^K$:

$$\begin{bmatrix} -\Xi(\tilde{R}) & \tilde{A} + \tilde{B}_2 \tilde{D}^K & \tilde{B}_1 & 0 \\ (\tilde{A} + \tilde{B}_2 \tilde{D}^K)^* & -\tilde{R}^{-1} & 0 & (\tilde{C}_1 + \tilde{D}_{12} \tilde{D}^K)^* \\ \tilde{B}_1^* & 0 & -I & \tilde{D}_{11}^* \\ 0 & \tilde{C}_1 + \tilde{D}_{12} \tilde{D}^K & \tilde{D}_{11} & -I \end{bmatrix} < 0, \tag{18}$$

and the state-feedback control law would then be $u = D^K y$.

Alternatively, instead of first trying to find R and then, if successful, solving for D^K, we may lump both of these steps into one, as shown in the following theorem.

Theorem 15: *Suppose that plant G is (h,q)-eventually periodic with $C_2 = I$ and $D_{21} = 0$. Then there exists an admissible (h,q)-eventually periodic static synthesis for G if and only if there exist block-diagonal matrices $\tilde{R} > 0$ and $\tilde{Q}$ satisfying*

$$\begin{bmatrix} -\Xi(\tilde{R}) & \tilde{A}\tilde{R}+\tilde{B}_2\tilde{Q} & \tilde{B}_1 & 0 \\ (\tilde{A}\tilde{R}+\tilde{B}_2\tilde{Q})^* & -\tilde{R} & 0 & (\tilde{C}_1\tilde{R}+\tilde{D}_{12}\tilde{Q})^* \\ \tilde{B}_1^* & 0 & -I & \tilde{D}_{11}^* \\ 0 & \tilde{C}_1\tilde{R}+\tilde{D}_{12}\tilde{Q} & \tilde{D}_{11} & -I \end{bmatrix} < 0.$$

If the above problem is feasible, then $\tilde{D}^K = \tilde{Q}\tilde{R}^{-1}$.

The proof consists of pre- and post-multiplying (18) by $\mathrm{diag}(I, \tilde{R}, I, I)$, and then setting $\tilde{Q} = \tilde{D}^K \tilde{R}$.

Remark 16: *It is not difficult to construct counter examples to demonstrate that, if we are to include sensor noise, i.e. $D_{21} \neq 0$, then the BSC might not admit (h,q)-eventually periodic solutions in $\mathfrak{X}$ even when feasible.*

B. Related state and measurement disturbances

In this subsection, we consider (h,q)-eventually periodic plants in which the state disturbance is a linear transformation of the sensor noise at each instance of the finite horizon, i.e., for $i = 0,1,\ldots,h-1$, there exists $T_i \in \mathbb{R}^{n_{i+1}\times n_{yi}}$ such that $B_{1i} = T_i D_{21i}$. We will show that, for such systems, a solution in $\mathfrak{X}$ exists for the BSC if and only if an (h,q)-eventually periodic solution in $\mathfrak{X}$ exists. Furthermore, if we start with eventually q-periodic solutions for the FSC and BSC, satisfying the coupling condition (11), then we can construct from these solutions (h,q)-eventually periodic solutions that still satisfy the coupling condition.

Lemma 17: *Suppose that the relevant system operators are (h,q)-eventually periodic and block-diagonal, and that, for $i = 0,1,\ldots,h-1$, we have $B_{1i} = T_i D_{21i}$ for some $T_i \in \mathbb{R}^{n_{i+1}\times n_{yi}}$. Then there exists a solution in $\mathfrak{X}$ to the BSC if and only if there exists an (h,q)-eventually periodic solution in $\mathfrak{X}$.*

Proof: The proof of the "if" direction is immediate. We now prove the "only if" direction. By assumption, there exists a solution in $\mathfrak{X}$ satisfying the BSC, then, by Theorem 6, there exists an (N,q)-eventually periodic solution $S = \mathrm{diag}(S_0,\ldots,S_{N-1},\hat{S}_{per},\hat{S}_{per},\ldots) \in \mathfrak{X}$ for some $N \geq h$, where $\hat{S}_{per} = \mathrm{diag}(S_N,\ldots,S_{N+q-1})$. Now, recall that $C_{2i}U_{1i} + D_{21i}U_{2i} = 0$, and hence, for $i = 0,1,\ldots,h-1$, we have $B_{1i}U_{2i} = T_i D_{21i} U_{2i} = -T_i C_{2i} U_{1i}$. Then, the finite horizon part of the BSC can be equivalently rewritten as

$$U_{1i}^*\left(\left(A_i - T_i C_{2i}\right)^* S_{i+1}\left(A_i - T_i C_{2i}\right) - S_i\right)U_{1i} + W_i < 0,$$

for $i = 0,1,\ldots,h-1$. Now defining

$$\bar{S}_{h-1} = P_{h-1}^* S_N P_{h-1} + S_{h-1} > 0,$$

and, for $i = h-2, h-3, \ldots, 0$,

$$\bar{S}_i = P_i^* \bar{S}_{i+1} P_i + S_i > 0,$$

where $P_i = A_i - T_i C_{2i}$, it is quite obvious that the operator $S_{eper} = \mathrm{diag}(\bar{S}_0, \bar{S}_1, \ldots, \bar{S}_{h-1}, \hat{S}_{per}, \hat{S}_{per}, \ldots)$ is an (h,q)-eventually periodic solution in $\mathfrak{X}$ to the BSC. ■

Theorem 18: *Suppose that plant G is (h,q)-eventually periodic, and that, for all $i = 0,1,\ldots,h-1$, we have $B_{1i} = T_i D_{21i}$ for some $T_i \in \mathbb{R}^{n_{i+1}\times n_{yi}}$. Then there exists an admissible synthesis K for G with state dimension $r \leq n$ if and only if there exist block-diagonal matrices $\tilde{R} > 0$ and $\tilde{S} > 0$ satisfying (5), (6), and (7).*

The key point of the proof of this theorem is to realize that, given eventually q-periodic operators solving the FSC and BSC and satisfying the coupling condition, the way we construct (h,q)-eventually periodic solutions from the said operators, as demonstrated in the proofs of Theorem 12 and Lemma 17, allows to form such solutions while still maintaining the validity of the coupling condition.

Remark 19: *Suppose the existence of a synthesis. If the matrices D_{21i}, for $i = 0,1,\ldots,h-1$, have each full column rank, then the condition $B_{1i} = T_i D_{21i}$ becomes trivial, and an (h,q)-eventually periodic synthesis exists. Also, the case where the finite horizon matrices U_{1i} have each full column rank warrants (h,q)-eventually periodic syntheses.*

VI. Conclusions

In this paper, we have shown that, given an (h,q)-eventually periodic system, then there exists an admissible synthesis for this system if and only if there exists an eventually q-periodic synthesis. Furthermore, we have proven that the forward synthesis condition if feasible always admits an (h,q)-eventually periodic solution. But the same thing cannot be said for the backward synthesis condition; also, the coupling condition might pose some restrictions as well. Lastly, we have considered systems with exactly measurable states as well as systems with related state and measurement disturbances, and shown that, for such (h,q)-eventually periodic systems, the synthesis if existent can always be chosen to be (h,q)-eventually periodic.

References

[1] J. A. Ball, I. Gohberg and M. A. Kaashoek, "Nevanlinna-Pick interpolation for time-varying input-output maps: the discrete case," *Operator Theory: Advances and Applications*, 56:1–51, Birkhauser, Basel, 1992.

[2] G. E. Dullerud and S. G. Lall, "A new approach to analysis and synthesis of time-varying systems," *IEEE Transactions on Automatic Control*, 44:1486–1497, 1999.

[3] M. Farhood and G.E. Dullerud, "LMI tools for eventually periodic systems," *Systems and Control Letters*, 47:417–432, 2002.

[4] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H_∞ control," *International Journal of Robust and Nonlinear Control*, 4:421–448, 1991.

[5] M. Green and D.J.N. Limebeer, *Linear Robust Control*, Prentice Hall, 1995.

[6] A. Halanay and V. Ionescu, *Time-Varying Discrete Linear Systems*, Birkhäuser, 1994.

[7] P. A. Iglesias, "An entropy formula for time-varying discrete-time control systems," *SIAM Journal on Control and Optimization*, 34:1691–1706, 1996.

[8] T. Iwasaki and R. E. Skelton, "All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas," *Automatica*, 30:1307–1317, 1994.

[9] A. Packard, "Gain scheduling via linear fractional transformations," *Systems and Control Letters*, 22:79–92, 1994.