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Model reduction of periodic systems: a lifting approach☆

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Abstract

This note furthers existing results on the model reduction of stable *periodic* systems. It utilizes for that matter a lifting technique to potentially attain less conservative error bounds.

Keywords: Balanced truncation; Stable periodic systems; Lifted state space representation

1. Introduction

This note provides a supplementary method for the model reduction of stable periodic systems. The approach proposed herein differs from those of Lall, Beck, and Dullerud (1998), Lall and Beck (2003), Longhi and Orlando (1999), Sandberg and Rantzer (2004), and Varga (2000) in that balanced truncation techniques are applied to a time-invariant reformulation of the periodic system rather than the periodic realization itself. This method proves to be especially useful and clearly advantageous over the currently available techniques when the model in question belongs to the class of stable periodic systems of large number of states, moderate period lengths, and relatively small number of inputs and/or outputs. Specifically, for models of the aforementioned class, the error bounds supplied by the proposed approach are usually far smaller than those given by the previous methods, for the same order reduction.

Our treatment of periodic systems follows that of Lall et al. (1998), where such systems are shown to have a special structure that allows the model reduction problem to be reduced to a finite-dimensional one, namely that of reducing a linear time-invariant (LTI) model with an associated uncertainty description; the latter problem is tackled in Beck, Doyle, and Glover (1996). This note also utilizes the computational procedure given in Varga (2004) for computing the minimal periodic realization of a lifted state-space representation. We remark that the aforesaid procedure is an improved version of the algorithm presented in Lin and King (1993). Further note that the procedure of Lall et al., (1998), can be extended to the class of *eventually* periodic systems, defined and studied in Farhood and Dullerud (2002), to yield a finite sum error bound on the model reduction of such systems. The reader is referred to Hinrichsen and Pritchard (1990) and the references therein for results on the model reduction of standard LTI systems. As far as the notation is concerned, it is quite standard. The set of real $n \times m$ matrices is denoted by $\mathbb{R}^{n\times m}$. If S_i is a sequence of operators, then $\mathrm{diag}(S_i)$ denotes their block-diagonal augmentation. The adjoint of an operator X is written X^*. The normed space of square summable vector-valued sequences is denoted by ℓ_2. Last, the notation $\|P\|_{\ell_2\to\ell_2}$ designates the ℓ_2-induced norm of a bounded linear mapping P on ℓ_2.

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2. Periodic systems and balanced truncation

This section briefly reviews periodic state space systems à la Lall et al., (1998), and further discusses the balanced truncation of such systems. Let G be the discrete-time q-periodic input–output system defined by the following time-varying difference equations for all integers $k \geqslant 0$:

$$x_{k+1} = A_k x_k + B_k w_k, \quad x_0 = 0,$$
$$z_k = C_k x_k + D_k w_k, \tag{1}$$

where the matrices $A_k \in \mathbb{R}^{n_{k+1}\times n_k}$, $B_k \in \mathbb{R}^{n_{k+1}\times m_k}$, $C_k \in \mathbb{R}^{p_k\times n_k}$, $D_k \in \mathbb{R}^{p_k\times m_k}$ and the integers n_k, m_k, p_k are all periodic with positive integer period q. We define the block-diagonal matrix $\tilde{A} := \operatorname{diag}(A_0, A_1, \ldots, A_{q-1})$, and similarly define $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$. In the sequel, we will use the quartet $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ as a succinct representation of the periodic system realization. We define the cyclic shift matrix $\tilde{Z}$ for $q \geqslant 2$ by

$$\tilde{Z} = \begin{bmatrix} 0 & \cdots & 0 & I \\ I & \ddots & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix}, \text{ such that}$$

$$\tilde{Z}^*\tilde{A}\tilde{Z} = \begin{bmatrix} A_1 & & & 0 \\ & \ddots & & \\ & & A_{q-1} & \\ 0 & & & A_0 \end{bmatrix}.$$

For $q = 1$, we set $\tilde{Z} = I$.

It is possible to show that the spectral radius of $\tilde{Z}\tilde{A}$ is strictly less than one if and only if the system $x_{k+1} = A_k x_k$ is exponentially stable. Throughout the paper, we will say a periodic state space system is *stable* when its A-operator satisfies this condition. At this point, we define the set $\tilde{\mathscr{X}}$ to consist of the positive definite operators $\tilde{X}$ having the form $\tilde{X} = \operatorname{diag}(X_0, X_1, \ldots, X_{q-1})$, where each X_i is a positive definite matrix in $\mathbb{R}^{n_i\times n_i}$. Following is an important lemma from Lall et al., (1998).

Lemma 1. *The following are equivalent*:

(i) *the spectral radius of $\tilde{Z}\tilde{A}$ is strictly less than one*;
(ii) *there exists $\tilde{Y} \in \tilde{\mathscr{X}}$ such that*

$$\tilde{A}\tilde{Y}\tilde{A}^* - \tilde{Z}^*\tilde{Y}\tilde{Z} + \tilde{B}\tilde{B}^* < 0; \tag{2}$$

(iii) *there exists $\tilde{X} \in \tilde{\mathscr{X}}$ such that*

$$\tilde{A}^*\tilde{Z}^*\tilde{X}\tilde{Z}\tilde{A} - \tilde{X} + \tilde{C}^*\tilde{C} < 0. \tag{3}$$

It is worth noting that Dullerud and Lall (1999) gives a more general result than the above lemma, notably a version of the Kalman–Yakubovich–Popov (KYP) lemma for the case of periodic systems. Also, note that, in the above lemma, the non-unique solutions $\tilde{X}$ and $\tilde{Y}$ are usually referred to as *generalized gramians*.

We now give an explicit definition of a *balanced* realization for periodic systems in terms of generalized gramians. Note that such balanced realizations always exist for stable periodic systems, and they are non-unique.

Definition 2. The linear periodic system realization is described as balanced if there exists $\tilde{Y}, \tilde{X} \in \tilde{\mathscr{X}}$ satisfying inequalities (2) and (3), respectively, such that

$$\tilde{X} = \tilde{Y} = \tilde{\Sigma} = \operatorname{diag}(\Sigma_0, \Sigma_1, \ldots, \Sigma_{q-1}),$$

where the matrices Σ_i are diagonal and positive definite.

The following proposition stems from Lemma 1 and Definition 2.

Proposition 3. *Given a stable periodic system G with realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, then there exists a state space transformation $\tilde{T} = \operatorname{diag}(T_0, T_1, \ldots, T_{q-1})$, where $T_i \in \mathbb{R}^{n_i\times n_i}$, such that the equivalent realization $(\underline{\tilde{A}}, \underline{\tilde{B}}, \underline{\tilde{C}}, \tilde{D}) = ((\tilde{Z}^*\tilde{T}\tilde{Z})\tilde{A}\tilde{T}^{-1}, (\tilde{Z}^*\tilde{T}\tilde{Z})\tilde{B}, \tilde{C}\tilde{T}^{-1}, \tilde{D})$ is balanced; and hence, there exists a diagonal matrix $\tilde{\Sigma} \in \tilde{\mathscr{X}}$ satisfying*

$$\underline{\tilde{A}}\tilde{\Sigma}\underline{\tilde{A}}^* - \tilde{Z}^*\tilde{\Sigma}\tilde{Z} + \underline{\tilde{B}}\underline{\tilde{B}}^* < 0,$$
$$\underline{\tilde{A}}^*\tilde{Z}^*\tilde{\Sigma}\tilde{Z}\underline{\tilde{A}} - \tilde{\Sigma} + \underline{\tilde{C}}^*\underline{\tilde{C}} < 0.$$

One such state space transformation $\tilde{T}$ is given by $T_i = \Sigma_i^{1/2}U_i^*Y_i^{-1/2}$ for $i = 0, 1, \ldots, q-1$, where the unitary matrix U_i and the diagonal positive-definite matrix Σ_i are obtained by performing a singular value decomposition on the matrix $Y_i^{1/2}X_iY_i^{1/2}$, namely $U_i\Sigma_i^2U_i^* = Y_i^{1/2}X_iY_i^{1/2}$. Note that Y_i and X_i are the constituent matrices of $\tilde{Y}$ and $\tilde{X}$ respectively, which, in turn, are solutions to inequalities (2) and (3).

We now discuss balanced truncation with guaranteed error bounds for periodic systems. We remark that these results are first introduced in Lall et al., (1998). Consider a stable system G with a balanced realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. As this realization is balanced, both of the Lyapunov inequalities (2) and (3) admit a common solution in $\tilde{\mathscr{X}}$, say $\tilde{\Sigma}$, which is of the form $\tilde{\Sigma} = \operatorname{diag}(\Sigma_0, \Sigma_1, \ldots, \Sigma_{q-1}) > 0$. We assume without loss of generality that, in each block Σ_i, the singular values are ordered along the diagonal with the largest first. Now given the q-periodic integers r_i such that $0 \leqslant r_i \leqslant n_i$ for all $i \geqslant 0$, we partition each of the Σ_i blocks into two sub-blocks $\Gamma_i \in \mathbb{R}^{r_i\times r_i}$ and $\Omega_i \in \mathbb{R}^{(n_i-r_i)\times(n_i-r_i)}$ so that $\Sigma_i = \operatorname{diag}(\Gamma_i, \Omega_i)$. Note that, since r_i is allowed to be equal to zero or n_i for each i, we are likely to have matrices with zero dimensions, which are ill-favored mathematically. However, such abuse of notation is adopted so as to allow for the possibilities that either zero states or all states are truncated at a particular time. We define the matrices $\tilde{\Gamma}$ and $\tilde{\Omega}$ to have a similar structure to that of $\tilde{\Sigma}$, namely $\tilde{\Gamma} = \operatorname{diag}(\Gamma_0, \Gamma_1, \ldots, \Gamma_{q-1})$ and $\tilde{\Omega} = \operatorname{diag}(\Omega_0, \Omega_1, \ldots, \Omega_{q-1})$. The singular values corresponding to the states that will be truncated are in $\tilde{\Omega}$.

At this point, we partition A_i, B_i and C_i conformably with the partitioning of Σ_i so that

$$A_i = \begin{bmatrix} \hat{A}_i & A_{i_{12}} \\ A_{i_{21}} & A_{i_{22}} \end{bmatrix}, \quad B_i = \begin{bmatrix} \hat{B}_i \\ B_{i_2} \end{bmatrix}, \quad \text{and} \quad C_i = [\hat{C}_i \;\; C_{i_2}].$$

Then the state space realization for the balanced truncation G_r of the system G is $(\tilde{A}_r, \tilde{B}_r, \tilde{C}_r, \tilde{D})$, where

$$\tilde{A}_r = \mathrm{diag}(\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_{q-1}),$$
$$\tilde{B}_r = \mathrm{diag}(\hat{B}_0, \hat{B}_1, \ldots, \hat{B}_{q-1}),$$
$$\tilde{C}_r = \mathrm{diag}(\hat{C}_0, \hat{C}_1, \ldots, \hat{C}_{q-1}).$$

The following theorem gives an error bound for such a reduction process.

Theorem 4. *Suppose the periodic system G is stable and balanced, and let G_r be the reduced order model formed by truncating G. Then G_r is also stable and balanced, and further satisfies the norm condition*

$$\|G - G_r\|_{\ell_2 \to \ell_2} < 2(\omega_1 + \cdots + \omega_s), \tag{4}$$

where the ω_i are the distinct *eigenvalues of $\tilde{\Omega}$.*

This result is an improved version of its counterpart in Lall et al., (1998), and can be deduced from Lall and Beck (2003); an alternative proof is given in Sandberg and Rantzer (2004). Note that, when $q = 1$, in which case the system G is LTI, then the above theorem reduces to the standard LTI result (Glover, 1984; Hinrichsen & Pritchard, 1990) on the balanced truncation method of model reduction.

Remark 5. Given a balanced periodic realization $(\tilde{A}, \tilde{B}, \tilde{C})$, if $\tilde{\Sigma}$ is a solution to both of Lyapunov inequalities (2) and (3), then so are $\alpha\tilde{\Sigma}$ for all $\alpha \geqslant 1$ (as a simple example). Since the error bound in (4) is clearly dependent on our choice of $\tilde{\Sigma}$, such a choice should satisfy some criterion that yields reasonable error bounds, for example, selecting the solution with the minimum trace.

3. Model reduction via time-invariant reformulation

The current approach to model reduction of periodic systems differs from those adopted in Lall et al., (1998), Lall and Beck (2003), Longhi and Orlando (1999), Sandberg and Rantzer (2004), and Varga (2000) in that the balanced truncation techniques are applied to the time-invariant reformulation rather than the periodic model itself. The approach now presented lacks some of the direct flexibility of the method of Section 2; however, in certain cases, it provides significantly better truncation bounds. Specifically, in the method of Section 2, one can choose any number of available states to truncate at each instance in the period and then calculate an upper bound on the error incurred in the reduction process. The approach now presented allows such a flexibility in the number of states to be truncated at one point of the period, but, as we will show, the number of states at other points in time can simultaneously be reduced with *no* additional incurred mismatch with the original system. We will provide explicit bounds on the dimensions at these points, which are completely independent of the generalized gramians of the system. For certain classes of stable periodic systems, the new method proposed herein will typically give much smaller error bounds than those produced by the approach of Section 2 for the same number of truncated states.

As one of our objectives is to compare and contrast the current reduction technique with that highlighted in Section 2, we assume herein that the periodic model in question has a minimal realization. However it is important to note that the theory of this section holds irrespective of this assumption. Concerning the construction of minimal realizations for periodic systems, we refer the reader to Varga (2004), which gives a significantly improved version of the computational procedure first introduced in Lin and King (1993). The current section is divided into two subsections. The first presents the time-invariant reformulation and establishes some properties that link this LTI system with its corresponding periodic system. The second subsection focuses on model reduction of the aforementioned LTI system and obtaining the associated minimal periodic realization.

3.1. Time-invariant reformulation

Suppose that $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is a state space minimal realization of the periodic system G. We plan to analyze this periodic realization by means of an equivalent linear time-invariant representation. Noting that the periodic system transition matrix $\phi_A(k, k_0) := A_{k-1}A_{k-2}\cdots A_{k_0}$ for $k > k_0$ with $\phi_A(k_0, k_0) := I$, we fix κ to be some integer in the set $\{0, 1, \ldots, q-1\}$, and define $\bar{A}_\kappa = \phi_A(\kappa + q, \kappa)$,

$$\bar{B}_\kappa = [\phi_A(\kappa+q, \kappa+1)B_\kappa \;\; \cdots \;\; \phi_A(\kappa+q, \kappa+q-1)B_{\kappa+q-2} \;\; B_{\kappa+q-1}],$$

$$\bar{C}_\kappa = \begin{bmatrix} C_\kappa \\ C_{\kappa+1}\phi_A(\kappa+1, \kappa) \\ \vdots \\ C_{\kappa+q-1}\phi_A(\kappa+q-1, \kappa) \end{bmatrix},$$

$$\bar{D}_\kappa = \begin{bmatrix} D_\kappa & & & 0 \\ F_{\kappa,2,1} & D_{\kappa+1} & & \\ \vdots & \vdots & \ddots & \\ F_{\kappa,q,1} & F_{\kappa,q,2} & \cdots & D_{\kappa+q-1}, \end{bmatrix},$$

where $F_{\kappa,i,j} = C_{\kappa+i-1}\phi_A(\kappa+i-1, \kappa+j)B_{\kappa+j-1}$. Now consider the time-invariant system $\bar{G}_\kappa$ described by the following equations for all integers $t \geqslant 0$:

$$\bar{x}_\kappa(t+1) = \bar{A}_\kappa \bar{x}_\kappa(t) + \bar{B}_\kappa \bar{w}_\kappa(t), \quad \bar{x}_\kappa(0) = x_\kappa,$$
$$\bar{z}_\kappa(t) = \bar{C}_\kappa \bar{x}_\kappa(t) + \bar{D}_\kappa \bar{w}_\kappa(t), \tag{5}$$

where $\bar{x}_\kappa(t) = x_{\kappa+tq}$, and

$$\bar{w}_\kappa(t) = [w^*_{\kappa+tq} \ \cdots \ w^*_{\kappa+(t+1)q-1}]^*,$$
$$\bar{z}_\kappa(t) = [z^*_{\kappa+tq} \ \cdots \ z^*_{\kappa+(t+1)q-1}]^*.$$

This time-invariant reformulation, often called the *lifted system* at time κ, can be viewed as a state-sampled representation of its corresponding periodic system with augmented input and output vectors. Note that since the periodic realization in question is minimal, then so is the lifted state space representation. It is obvious that the stability of G is equivalent to the stability of $\bar{G}_\kappa$. Also, the periodic triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ is stabilizable and detectable if and only if the time-invariant triplet $(\bar{A}_\kappa, \bar{B}_\kappa, \bar{C}_\kappa)$ is stabilizable and detectable. The definitions of the stabilizability and detectability of discrete-time periodic systems as well as the proof of the preceding structural property and others can be found in Bittanti, Colaneri, and De Nicolao (1986, 1991) and Bolzern and Colaneri (1987).

Clearly, an LTI system is a periodic system with period $q = 1$. Hence, the results and definitions of Section 2 apply here, where those results reduce to the by-now standard results of Glover (1984) and Hinrichsen and Pritchard (1990). At this point, we need to define additional mappings. Let $\mathbb{P}^e$ denote the set of positive semi-definite matrices of dimension e. Recalling that the integer sequence n_i gives the state dimensions of the periodic system, we define the mappings $\Lambda_i^b : \mathbb{P}^{n_{i+1}} \to \mathbb{P}^{n_i}$ and $\Lambda_i^f : \mathbb{P}^{n_i} \to \mathbb{P}^{n_{i+1}}$ by

$$\Lambda_i^b(P) = A_i^* P A_i + C_i^* C_i,$$
$$\Lambda_i^f(P) = A_i P A_i^* + B_i B_i^*,$$

respectively, for $i = 0, \ldots, q-1$. Furthermore, we define the mappings $\bar{\Lambda}_\kappa^b : \mathbb{P}^{n_\kappa} \to \mathbb{P}^{n_\kappa}$ and $\bar{\Lambda}_\kappa^f : \mathbb{P}^{n_\kappa} \to \mathbb{P}^{n_\kappa}$ by

$$\bar{\Lambda}_\kappa^b(P) = \bar{A}_\kappa^* P \bar{A}_\kappa + \bar{C}_\kappa^* \bar{C}_\kappa,$$
$$\bar{\Lambda}_\kappa^f(P) = \bar{A}_\kappa P \bar{A}_\kappa^* + \bar{B}_\kappa \bar{B}_\kappa^*.$$

We now state the following results.

Proposition 6. *The following are valid for all* $i = 0, 1, \ldots, q-1$:

(i) *If* $X, Y \in \mathbb{P}^{n_i}$ *and* $X \leqslant Y$, *then* $\Lambda_u^b(X) \leqslant \Lambda_u^b(Y)$ *and* $\Lambda_i^f(X) \leqslant \Lambda_i^f(Y)$, *where* $u := i - 1 + q \bmod q$.

(ii) *If the diagonal matrix* $\tilde{\Sigma} = \mathrm{diag}(\Sigma_0, \ldots, \Sigma_{q-1}) \in \tilde{\mathscr{X}}$ *satisfies both inequalities* (2) *and* (3), *then* $\Lambda_i^b(\Sigma_v) < \Sigma_i$ *and* $\Lambda_i^f(\Sigma_i) < \Sigma_v$, *where* $v := i + 1 \bmod q$.

Proof. To prove the first inequality in (i), note that, since the sequence n_k is q-periodic, the mapping Λ_u^b is well-defined on $\mathbb{P}^{n_i}$. Then, since $X \leqslant Y$, we have $A_u^* X A_u \leqslant A_u^* Y A_u$, and so $A_u^* X A_u + C_u^* C_u \leqslant A_u^* Y A_u + C_u^* C_u$ follows. A similar argument shows the second inequality.

Part (ii) follows by noting that both inequalities (2) and (3) are block-diagonal with the required inequalities in each of the blocks. □

Theorem 7. *Suppose* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *is a balanced periodic realization with a diagonal generalized gramian* $\tilde{\Sigma} = \mathrm{diag}(\Sigma_0, \ldots, \Sigma_{q-1}) \in \tilde{\mathscr{X}}$ *satisfying both Lyapunov inequalities* (2) *and* (3). *Then the corresponding lifted LTI system realization at time* κ *is also balanced and furthermore admits a common solution to both of its Lyapunov inequalities equal to* Σ_κ.

Proof. We need to prove that the inequalities $\bar{\Lambda}_\kappa^b(\Sigma_\kappa) < \Sigma_\kappa$ and $\bar{\Lambda}_\kappa^f(\Sigma_\kappa) < \Sigma_\kappa$ hold. To start, we can prove by induction that

$$\bar{\Lambda}_\kappa^b(\Sigma_\kappa) = \Lambda_\kappa^b(\Lambda_{\kappa+1}^b(\cdots \Lambda_{\kappa+q-1}^b(\Sigma_\kappa)\cdots)),$$
$$\bar{\Lambda}_\kappa^f(\Sigma_\kappa) = \Lambda_{\kappa+q-1}^f(\Lambda_{\kappa+q-2}^f(\cdots \Lambda_\kappa^f(\Sigma_\kappa)\cdots)).$$

Then, invoking part (ii) of Proposition 6 together with an iterative application of part (i) leads to the desired inequalities. □

3.2. Model reduction of periodic systems

Suppose that $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is a balanced realization of the periodic system G with generalized gramians $\tilde{X}$ and $\tilde{Y}$ such that $\tilde{X} = \tilde{Y} = \tilde{\Sigma} = \mathrm{diag}(\Sigma_0, \ldots, \Sigma_{q-1}) \in \tilde{\mathscr{X}}$, where $\tilde{\Sigma}$ is a diagonal matrix. Then, by Theorem 7, the realization of the lifted LTI system $\bar{G}_\kappa$ is also balanced with diagonal generalized gramian Σ_κ satisfying both of its corresponding Lyapunov inequalities. Recall the assumption that the singular values of Σ_κ are ordered along the diagonal with the largest first. Following the balanced truncation procedure of Section 2 (which in this LTI case reduces exactly to that of Glover (1984) and Hinrichsen & Pritchard (1990)) and the notation established there, given r_κ such that $0 < r_\kappa < n_\kappa$, we partition $\Sigma_\kappa \in \mathbb{R}^{n_\kappa \times n_\kappa}$ into two sub-blocks $\Gamma_\kappa \in \mathbb{R}^{r_\kappa \times r_\kappa}$ and $\Omega_\kappa \in \mathbb{R}^{(n_\kappa - r_\kappa)\times(n_\kappa - r_\kappa)}$ so that $\Sigma_\kappa = \mathrm{diag}(\Gamma_\kappa, \Omega_\kappa)$. The singular values corresponding to the states that will be truncated are in Ω_κ.

Partitioning the system matrices of $\bar{G}_\kappa$ conformably with the partitioning of Σ_κ, we get

$$\left[\begin{array}{c|c} \bar{A}_\kappa & \bar{B}_\kappa \\ \hline \bar{C}_\kappa & \bar{D}_\kappa \end{array}\right] = \left[\begin{array}{cc|c} \bar{A}_{\kappa,r} & \bar{A}_{\kappa,12} & \bar{B}_{\kappa,r} \\ \bar{A}_{\kappa,21} & \bar{A}_{\kappa,22} & \bar{B}_{\kappa,2} \\ \hline \bar{C}_{\kappa,r} & \bar{C}_{\kappa,2} & \bar{D}_\kappa \end{array}\right].$$

Then the state space realization for the balanced truncation $\bar{G}_{\kappa,r}$ of the system $\bar{G}_\kappa$ is $(\bar{A}_{\kappa,r}, \bar{B}_{\kappa,r}, \bar{C}_{\kappa,r}, \bar{D}_\kappa)$. Clearly, $\bar{G}_{\kappa,r}$ is a balanced stable model, and furthermore, it satisfies the norm condition

$$\|\bar{G}_\kappa - \bar{G}_{\kappa,r}\|_{\ell_2\to\ell_2} < 2(\omega_{\kappa,1} + \cdots + \omega_{\kappa,v}),$$

where $\omega_{\kappa,i}$ are the *distinct* eigenvalues of Ω_κ. Now let G_r be the reduced periodic system corresponding to the LTI system $\bar{G}_{\kappa,r}$. Noting that Σ_κ is positive *definite*, it is then immediate that G_r is stable and satisfies

$$\|G - G_r\|_{\ell_2\to\ell_2} = \|\bar{G}_\kappa - \bar{G}_{\kappa,r}\|_{\ell_2\to\ell_2} < 2(\omega_{\kappa,1} + \cdots + \omega_{\kappa,v}).$$

Having established this, the next step is to obtain the minimal realization of the reduced periodic system G_r. This problem, however, has been tackled in Varga (2004), where an improved algorithm to that of Lin and King (1993) is proposed for computing the minimal periodic realization of a lifted state-space representation. We refrain from reviewing the aforementioned computational procedure, and restrict the following presentation to the discussion of the dimensions of the reduced periodic model. We start by partitioning the truncated lifted state-space matrices in accordance with the partitioning of the augmented input and output vectors so that

$$\bar{B}_{\kappa,r} = [\bar{B}^0_{\kappa,r}\ \bar{B}^1_{\kappa,r}\ \cdots\ \bar{B}^{q-1}_{\kappa,r}],$$

$$\bar{C}_{\kappa,r} = \begin{bmatrix} \bar{C}^0_{\kappa,r} \\ \bar{C}^1_{\kappa,r} \\ \vdots \\ \bar{C}^{q-1}_{\kappa,r} \end{bmatrix}, \quad \bar{D}_\kappa = \begin{bmatrix} \bar{D}^{0,0}_\kappa & & 0 \\ \vdots & \ddots & \\ \bar{D}^{q-1,0}_\kappa & \cdots & \bar{D}^{q-1,q-1}_\kappa \end{bmatrix}.$$

Similarly Lin and King (1993), and Varga (2004), we define the following sequence of matrices for $i = 0, 1, \ldots, q-2$:

$$K_i = \begin{bmatrix} \bar{A}_{\kappa,r} & \bar{B}^0_{\kappa,r} & \cdots & \bar{B}^i_{\kappa,r} \\ \bar{C}^{q-1}_{\kappa,r} & \bar{D}^{q-1,0}_\kappa & & 0 \\ \vdots & \vdots & \ddots & \\ \bar{C}^{i+1}_{\kappa,r} & \bar{D}^{i+1,0}_\kappa & \cdots & \bar{D}^{i+1,i}_\kappa \end{bmatrix}.$$

Then, from Varga (2004), the q-periodic dimensions of the reduced periodic realization at all the points of the period excluding κ are given by $r_{\kappa+i+1} = \operatorname{rank}(K_i)$, for $i = 0, 1, \ldots, q-2$. The following upper bounds on the aforementioned dimensions are thus immediate:

$$r_{\kappa+i} \leqslant \min\left(r_\kappa + \sum_{j=\kappa}^{\kappa+i-1} m_j,\ r_\kappa + \sum_{j=\kappa+i}^{\kappa+q-1} p_j,\ n_{\kappa+i} \right),$$

for $i = 1, \ldots, q-1$. It is not difficult to show that the first term in the above min function can be replaced with the tighter bound $r_{\kappa+i-1} + m_{\kappa+i-1}$.

We now elucidate the rationale behind the model reduction method given in this section. Consider the class of stable periodic systems of moderate period lengths and large number of states at each instance in the period with relatively small number of inputs and/or outputs. Given a system of such a class, we can always represent it with a balanced periodic realization, and furthermore, find a diagonal generalized gramian $\tilde{\Sigma}$ that satisfies some criterion, as argued in Remark 5, and solves both Lyapunov inequalities (2) and (3). We then determine the Σ_i block with the minimum "sum of the tail;" say such block corresponds to the instance κ in the period, where κ is a fixed integer in the set $\{0, 1, \ldots, q-1\}$. We may also further improve this minimum by re-solving the aforesaid Lyapunov inequalities for the diagonal solution with the minimum trace of Σ_κ. Afterwards, we compute the lifted LTI system at κ, as shown in the first subsection of this section, and then we implement the balanced truncation method of model reduction on this LTI model and obtain the minimal periodic realization corresponding to the truncated lifted representation, as illustrated in the second subsection. For such a system, the reduction in the model dimensions is significant not only at the point κ but also at all the other points in the period. Furthermore, the associated error bound can be of a much smaller value than that calculated for the same order reduction by the approach in Section 2. Consequently, the method herein serves as a better guideline for the model reduction of stable periodic systems of the aforementioned class.

Remark 8. In Definition 2, we make use of *generalized* gramians that satisfy the *strict* Lyapunov inequalities (2)and (3) to define a notion of a balanced realization for periodic systems. However, given a stable periodic system, the lifting approach of this paper and particularly Theorem 7 still apply if instead the definition of a balanced realization is given in terms of the standard controllability and observability gramians which are solutions of Lyapunov equations or generalized gramians solving non-strict Lyapunov inequalities. Note that as the positive definiteness of the gramians is necessary, the minimality of the periodic system becomes a requirement when using standard gramians; also, in such a case, the truncation of a balanced system is not necessarily balanced.

4. Examples

It is not difficult to construct numerical examples where the current approach has clear advantages over the standard approaches of Lall et al., (1998), Lall and Beck (2003), Longhi and Orlando (1999), Sandberg and Rantzer (2004), and Varga (2000). Specifically, we have written a MATLAB code that generates random, stable, minimal and balanced periodic SISO system realizations of user-specified period lengths and constant state orders. Also, by specifying the allowable reduction error as a percentage of the system norm, the program locates the point of the period with the minimum sum-of-the-tail, lifts the system at this point to obtain the LTI reformulation, applies balanced truncation, and then "unlifts" the truncated LTI system to get the reduced periodic realization. Other data provided by the code are the actual reduction error and the upper error bound, together with their counterparts given by the standard approaches for the same order reduction. The algorithm used for obtaining stable and balanced minimal realizations is as follows: First, choose random system matrices of appropriate dimensions such that the A-matrices are all diagonal and invertible with singular values ranging from 0.16–0.96; this guarantees stability and minimality for moderate period lengths. Then, solve for the gramians satisfying the Lyapunov equalities; the fact that $\|\tilde{A}\| < 1$, and hence $\tilde{A}^*\tilde{A} < I$, allows for the perturbation of these gramians by εI for some sufficiently small positive ε to avoid any numerical inaccuracies while still maintaining valid Lyapunov inequalities. Note that since the theory herein is applicable regardless of the definition of balanced realizations (see Remark 8), then the use of both Lyapunov equalities and inequalities is acceptable. Lastly, using the Cholesky factors of these gramians, we derive a balanced realization. Of course the diagonal structure of the A-matrices is generally lost after balancing. This MATLAB code can be found at http://legend.me.uiuc.edu/∼mazen/Lifting/. All examples generated by this code are generally in favor of the lifting approach. One example that we give at this Web site pertains to a SISO system of ℓ_2-induced norm equal to 18.7, period length equal to 10, and with 30 states at each point of the period. For a reduced periodic model of dimensions $(r_i)_{i=0}^{9} = 1, 2, 3, 4, 5, 6, 5, 4, 3, 2$, the following table displays the actual reduction error and upper error bound given by the lifting and standard approaches, where the balanced realizations are obtained as described above.

Lifting approach		Standard approach	
$\|G - G_r\|$	Error bound	$\|G - G_r\|$	Error bound
0.0628	0.3281	0.0630	1.2367

Clearly, the lifting approach gives a much smaller error bound. Note that in this example, the actual errors from both approaches are roughly equal, but this is obviously not true in general since each approach typically yields a different reduced model.

5. Conclusions

This paper provides a lifting approach for the model reduction of stable periodic systems. In the case of moderate period lengths, large number of states, and relatively small number of inputs and/or outputs, the error bounds given by this approach are typically much smaller than those supplied by the currently available balanced truncation methods for the same order reduction.

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