Model Reduction of Strongly Stable Nonstationary LPV Systems

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1 Introduction

This paper deals with the model reduction of nonstationary linear parameter-varying (NLPV) systems. Our interest in LPV models is motivated by the desire to control nonlinear systems along prespecified trajectories. LPV models arise naturally in such scenarios as a method to capture the possible nonlinear dynamics, while maintaining a model that is amenable to control synthesis. Frequently, when pursuing such an LPV formulation, one ends up with models of relatively large dimension. Accordingly, finding control syntheses for such models, which usually involves solving a number of linear operator inequalities as discussed in [5], requires substantial computation. For this reason, developing a theory that provides systematic methods of approximating such models is beneficial.

In the paper, we utilize the theory of generalized gramians to define the notion of balanced realizations for NLPV systems. We also examine the balanced truncation method in detail and derive error bounds for such a reduction process. The contributions of the paper are as follows:

- generalization of the balanced truncation model reduction procedure to the class of NLPV systems;
- several results on the worst-case balanced truncation error. These results when restricted to the purely time-varying case (i.e., no parameters) provide the least conservative error bounds currently available in the literature;
- operator theoretic machinery is developed in the context of standard robust control tools for working with NLPV models.

Our paper deploys a combination of recent work on NLPV models in [5] and new work on model reduction using balanced truncation for standard LTV systems in [13, 18]. The basic approach is motivated by the work in [1] on the generalization of balanced truncation to stationary multidimensional systems, and that in [12] on discrete time model reduction of standard LTI

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systems. The basic approach behind balanced truncation originates in [16], and the by-now famous error bounds associated with this method in the LTI case were first demonstrated in [3, 11]. The NLPV models used here are the natural generalization of LPV models, first introduced in [15, 17], to the case of nonstationary systems.

The paper is organized as follows: we begin with a section which establishes notation and collects some needed definitions; in Section 3 we introduce NLPV models and discuss two forms of stability; Section 4 provides the balanced truncation procedure and proves the main results of the paper; we conclude with a summary statement.

2 Preliminaries

The set of real $n \times m$ matrices and that of real symmetric $n \times n$ matrices are denoted by $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n$ respectively. The maximum singular value of a matrix X is denoted by $\bar{\sigma}(X)$.

Given two Hilbert spaces E and F, we denote the space of bounded linear operators mapping E to F by $\mathcal{L}(E, F)$, and shorten this to $\mathcal{L}(E)$ when E equals F. If X is in $\mathcal{L}(E, F)$, we denote the E to F induced norm of X by $\|X\|_{E\to F}$; when the spaces involved are obvious, we write simply $\|X\|$. The adjoint of X is written X^*. When an operator $X \in \mathcal{L}(E)$ is self-adjoint, we use $X \prec 0$ to mean it is negative definite; that is there exists a number $\alpha > 0$ such that, for all nonzero $x \in E$, the inequality $\langle x, Xx\rangle < -\alpha\|x\|^2$ holds, where $\langle\cdot,\cdot\rangle$ denotes the inner product and $\|\cdot\|$ denotes the corresponding norm on E. We use $E\oplus F$ to denote the Hilbert space direct sum of E and F. If S_i is a sequence of operators, then $\mathrm{diag}(S_i)$ denotes their block-diagonal augmentation.

The main Hilbert space of interest in this paper is formed from an infinite sequence of Euclidean spaces $(\mathbb{R}^{n(0)}, \mathbb{R}^{n(1)}, \mathbb{R}^{n(2)}, \ldots)$, and is denoted by $\ell_2(\mathbb{R}^{n(0)}, \mathbb{R}^{n(1)}, \ldots)$, or just $\ell_2(\mathbb{R}^n)$ for short. It is defined as the subspace of the Hilbert space direct sum $\oplus_{k=0}^{\infty}\mathbb{R}^{n(k)}$ consisting of elements $x = (x(0), x(1), x(2), \ldots)$, with $x(k) \in \mathbb{R}^{n(k)}$, so that $\|x\|^2 = \sum_{k=0}^{\infty} x(k)^*x(k) < \infty$. The inner product of x, y in $\ell_2(\mathbb{R}^n)$ is hence defined as the sum $\langle x, y\rangle_{\ell_2} = \sum_{k=0}^{\infty} x(k)^*y(k)$. If the spatial dimensions $n(k)$ are either evident or irrelevant to the discussion, then the notation $\ell_2(\mathbb{R}^n)$ is abbreviated to ℓ_2. Also, we will use $\ell(\mathbb{R}^n)$ to denote $\oplus_{k=0}^{\infty}\mathbb{R}^{n(k)}$.

A key operator used in the paper is the unilateral shift Z, defined as follows:

$$\begin{aligned} Z: \quad \ell_2\left(\mathbb{R}^{n(1)}, \mathbb{R}^{n(2)}, \ldots\right) &\to \ell_2\left(\mathbb{R}^{n(0)}, \mathbb{R}^{n(1)}, \mathbb{R}^{n(2)}, \ldots\right) \\ (a(1), a(2), \ldots) &\overset{Z}{\longmapsto} (0, a(1), a(2), \ldots). \end{aligned}$$

Clearly this definition is extendable to ℓ, and in the sequel, we will not distinguish between these mappings. Given a time-varying dimension $n(k)$, we define the notation $I_{\ell_2}^n := \mathrm{diag}(I_{n(0)}, I_{n(1)}, I_{n(2)}, \ldots)$, where $I_{n(k)}$ is an $n(k)\times n(k)$ identity matrix.

Following the notation and approach in [2], we make the following definitions. First, we say a bounded linear operator Q mapping $\ell_2(\mathbb{R}^{m(0)}, \mathbb{R}^{m(1)}, \ldots)$ to $\ell_2(\mathbb{R}^{n(0)}, \mathbb{R}^{n(1)}, \ldots)$ is *block-diagonal* if there exists a sequence of matrices $Q(k)$ in $\mathbb{R}^{n(k)\times m(k)}$ such that, for all w, z, if $z = Qw$, then $z(k) = Q(k)w(k)$. Then Q has the representation $\operatorname{diag}(Q(0), Q(1), Q(2), \ldots)$. A *diagonal* operator is a block-diagonal operator where each of the matrix blocks is diagonal.

Suppose F, G, R and S are block-diagonal operators, and let A be a *partitioned* operator of the form

$$A = \begin{bmatrix} F & G \\ R & S \end{bmatrix}.$$

Then we define the following notation:

$$[\![A]\!] := \operatorname{diag}\left(\begin{bmatrix} F(0) & G(0) \\ R(0) & S(0) \end{bmatrix}, \begin{bmatrix} F(1) & G(1) \\ R(1) & S(1) \end{bmatrix}, \ldots \right),$$

which we call the *diagonal realization* of A. Clearly for any given operator A of this particular structure, $[\![A]\!]$ is simply A with the rows and columns permuted appropriately so that

$$[\![A]\!]_k = \begin{bmatrix} F(k) & G(k) \\ R(k) & S(k) \end{bmatrix}.$$

From this definition, it is easy to see that $[\![A + B]\!] = [\![A]\!] + [\![B]\!]$ and $[\![AC]\!] = [\![A]\!]\,[\![C]\!]$ hold for appropriately dimensioned operators, and similarly that $A \prec \beta I$ holds if and only if $[\![A]\!] \prec \beta I$, where β is a scalar. Namely, the $[\![\bullet]\!]$ operation is a homomorphism from partitioned operators with block-diagonal entries to block-diagonal operators.

3 NLPV Systems

We now briefly review NLPV models. The reader is referred to [5] for an in-depth treatment of the theory. To start, the NLPV models of this paper are of the form

$$\begin{aligned} x(k+1) &= A(\delta(k), k)x(k) + B(\delta(k), k)w(k) \\ z(k) &= C(\delta(k), k)x(k) + D(\delta(k), k)w(k), \end{aligned}$$

where $A(\cdot,\cdot)$, $B(\cdot,\cdot)$, $C(\cdot,\cdot)$, and $D(\cdot,\cdot)$ are matrix-valued functions that are known *a priori*. The variable k is time, and $\delta(k) := (\delta_1(k), \ldots, \delta_d(k))$ is a vector of real scalar parameters. In this paper, we are concerned only with the subclass of NLPV models satisfying the condition that the dependence of the matrix functions A, B, C, and D on the parameters δ_i is rational and given in terms of a feedback coupling. Such models are commonly referred to as LFT systems and are basically the straightforward generalization of the

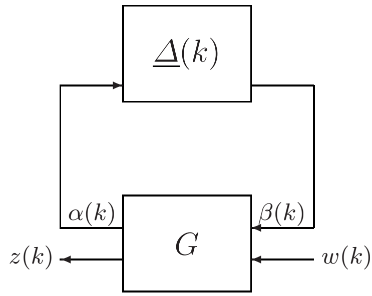


Fig. 1. The interconnection of G with $\underline{\Delta}(k)$

LPV systems first introduced in [15, 17]. We now introduce a model of the said subclass.

Let G be a linear time-varying discrete-time system defined by the following state space equation:

$$\begin{bmatrix} x(k+1) \\ \alpha(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_{ss}(k) & A_{sp}(k) & B_s(k) \\ A_{ps}(k) & A_{pp}(k) & B_p(k) \\ C_s(k) & C_p(k) & D(k) \end{bmatrix} \begin{bmatrix} x(k) \\ \beta(k) \\ w(k) \end{bmatrix}, \qquad x(0) = 0, \tag{1}$$

for $w \in \ell_2$. The vector-valued signals $x(k)$, $\alpha(k)$, $\beta(k)$, $z(k)$, and $w(k)$ are real and have time-varying dimensions, with the constraint that $\dim(\beta(k)) = \dim(\alpha(k))$. We denote the dimensions of these signals by $n_0(k)$, $n(k)$, $n(k)$, $n_z(k)$, and $n_w(k)$ respectively. We assume that all the state space matrices are uniformly bounded functions of time. For any scalar sequences $\delta_1(k), \ldots, \delta_d(k)$ and associated dimensions $n_1(k), \ldots, n_d(k)$ satisfying $\sum_{i=1}^{d} n_i(k) = n(k)$, we define the diagonal matrix $\underline{\Delta}(k)$ as

$$\underline{\Delta}(k) := \mathrm{diag}(\delta_1(k) I_{n_1(k)}, \ldots, \delta_d(k) I_{n_d(k)}) \in \mathbb{R}^{n(k) \times n(k)}.$$

Also, we also constrain $\bar{\sigma}(\underline{\Delta}(k)) \leq 1$ for all $k \geq 0$. We will be concerned with the feedback arrangement in Figure 1, where G and $\underline{\Delta}(k)$ are connected in feedback. This system can be expressed formally by

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = H(k) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}, \tag{2}$$

where $H(k)$ is given by

$$H(k) = \begin{bmatrix} A_{ss}(k) & B_s(k) \\ C_s(k) & D(k) \end{bmatrix} + \begin{bmatrix} A_{sp}(k) \\ C_p(k) \end{bmatrix} \underline{\Delta}(k) \left(I - A_{pp}(k) \underline{\Delta}(k)\right)^{-1} \begin{bmatrix} A_{ps}(k) \\ B_p(k) \end{bmatrix}. \tag{3}$$

We will refer to the mapping $w \mapsto z$ in (2) as the system G_δ. Hence, G_δ is a linear time-varying system with arbitrary rational state-space parameter dependence formulated in an LFT framework, where the time-varying parameters δ_i act on the system G through the linear fractional feedback channels

(α, β). We assume $A_{pp}(k)$ such that $S_k = I - A_{pp}(k)\underline{\Delta}(k)$ is invertible for all $k \geq 0$ so that the LFT in (3) is well-defined at each time k. This well-posedness condition guarantees that there are unique solutions in ℓ to (1).

Using the previously defined notation, clearly the matrix sequences $A_{ss}(k)$, $B_s(k)$, $C_s(k)$, and $D(k)$ from (1) define block-diagonal operators. The blocks of the matrix $\underline{\Delta}(k)$ naturally partition $\alpha(k)$ and $\beta(k)$ into d separate vector-valued channels, conformably with which we partition the following state space matrices such that

$$\begin{aligned} A_{sp}(k) &= \begin{bmatrix} A_{sp}^1(k) \; A_{sp}^2(k) \; \cdots \; A_{sp}^d(k) \end{bmatrix} \\ A_{pp}(k) &= \begin{bmatrix} A_{pp}^{11}(k) & \cdots & A_{pp}^{1d}(k) \\ \vdots & \ddots & \vdots \\ A_{pp}^{d1}(k) & \cdots & A_{pp}^{dd}(k) \end{bmatrix} \\ A_{ps}(k) &= \begin{bmatrix} A_{ps}^1(k) \\ A_{ps}^2(k) \\ \vdots \\ A_{ps}^d(k) \end{bmatrix} \qquad B_p(k) = \begin{bmatrix} B_p^1(k) \\ B_p^2(k) \\ \vdots \\ B_p^d(k) \end{bmatrix} \\ C_p(k) &= \begin{bmatrix} C_p^1(k) \; C_p^2(k) \; \cdots \; C_p^d(k) \end{bmatrix}, \end{aligned} \tag{4}$$

where $A_{sp}^i(k) \in \mathbb{R}^{n_0(k+1) \times n_i(k)}$, $A_{pp}^{ij}(k) \in \mathbb{R}^{n_i(k) \times n_j(k)}$, $A_{ps}^i(k) \in \mathbb{R}^{n_i(k) \times n_0(k)}$, $B_p^i(k) \in \mathbb{R}^{n_i(k) \times n_w(k)}$, and $C_p^i(k) \in \mathbb{R}^{n_z(k) \times n_i(k)}$. The matrix sequence of each of the elements of the state space matrices in (4) defines a block-diagonal operator; and so we construct from the sequence of each of these state space matrices a partitioned operator, each of whose elements is block-diagonal and defined in the obvious way. For instance, the matrix sequences $A_{sp}^1(k), \ldots, A_{sp}^d(k)$ define block-diagonal operators that compose the partitioned operator A_{sp}. With Z being the shift, we rewrite our system equations as

$$\begin{bmatrix} x \\ \alpha \\ z \end{bmatrix} = \begin{bmatrix} ZA_{ss} & ZA_{sp} & ZB_s \\ A_{ps} & A_{pp} & B_p \\ C_s & C_p & D \end{bmatrix} \begin{bmatrix} x \\ \beta \\ w \end{bmatrix}, \tag{5}$$

$$\begin{bmatrix} x \\ \beta \end{bmatrix} = \operatorname{diag}(I_{\ell_2}^{n_0}, \Delta_1, \ldots, \Delta_d) \begin{bmatrix} x \\ \alpha \end{bmatrix} = \Delta \begin{bmatrix} x \\ \alpha \end{bmatrix}, \tag{6}$$

where $x \in \ell(\mathbb{R}^{n_0})$, $w \in \ell_2(\mathbb{R}^{n_w})$, $z \in \ell(\mathbb{R}^{n_z})$, $\beta = (\beta_1, \ldots, \beta_d)$, $\alpha = (\alpha_1, \ldots, \alpha_d)$, β_i, $\alpha_i \in \ell(\mathbb{R}^{n_i})$, and

$$\Delta_i = \operatorname{diag}(\delta_i(0)I_{n_i(0)}, \delta_i(1)I_{n_i(1)}, \delta_i(2)I_{n_i(2)}, \ldots).$$

Before concluding this section, we make some convenient definitions that will be used extensively in the sequel. To start, we define $\tilde{Z} := \operatorname{diag}(Z, I_{\ell_2})$ and

$$A := \begin{bmatrix} A_{ss} & A_{sp} \\ A_{ps} & A_{pp} \end{bmatrix}, \; B := \begin{bmatrix} B_s \\ B_p \end{bmatrix}, \; C := \begin{bmatrix} C_s \; C_p \end{bmatrix}. \tag{7}$$

Also, we define $\ell_2^{(n_0,\ldots,n_d)} := \ell_2(\mathbb{R}^{n_0}) \oplus \ell_2(\mathbb{R}^{n_1}) \oplus \cdots \oplus \ell_2(\mathbb{R}^{n_d})$ and

$$\boldsymbol{\Delta} := \{\Delta \in \mathcal{L}(\ell_2^{(n_0,\ldots,n_d)}) : \Delta \text{ is partitioned as in (6) and } \|\Delta\| \leq 1\}.$$

Note that the operator $\tilde{Z} = \text{diag}(Z, I_{\ell_2})$ has a conformable partitioning to that of $\Delta = \text{diag}(\Delta_s, \Delta_p)$, where $\Delta_s = I_{\ell_2}^{n_0}$ and $\Delta_p = \text{diag}(\Delta_1, \Delta_2, \ldots, \Delta_d)$.

3.1 Stability of NLPV Models

This section tackles the various concepts of stability that are essential to our work. To begin we define a basic notion of stability.

Definition 21 *An NLPV model is ℓ_2-stable if $I - \Delta\tilde{Z}A$ has a bounded inverse for all $\Delta \in \boldsymbol{\Delta}$.*

Thus, there exists a unique $(x, \beta) \in \ell_2^{(n_0,\ldots,n_d)}$ satisfying (5) and (6) if the model is ℓ_2-stable. In such a scenario, equations (5) and (6) can be rewritten in the form $z = G_\delta w$, where

$$G_\delta = C(I - \Delta\tilde{Z}A)^{-1}\Delta\tilde{Z}B + D \in \mathcal{L}(\ell_2(\mathbb{R}^{n_w}), \ell_2(\mathbb{R}^{n_z})). \tag{8}$$

At this point, we define $\mathcal{T}$ as the set of operators $T \in \mathcal{L}(\ell_2^{(n_0,\ldots,n_d)})$ that have bounded inverses and are of the form $T = \text{diag}(T_0, T_1, \ldots, T_d)$, where each $T_i \in \mathcal{L}(\ell_2(\mathbb{R}^{n_i}))$ is block-diagonal so that $[\![T_i]\!]_k = T_i(k) \in \mathbb{R}^{n_i(k) \times n_i(k)}$. Observe that $\mathcal{T}$ is a commutant of $\boldsymbol{\Delta}$. Moreover, we define the subset $\mathcal{X}$ of $\mathcal{T}$ by $\mathcal{X} = \{X \succ 0 : X \in \mathcal{T}\}$.

Definition 22 *An NLPV model is strongly ℓ_2-stable if there exists $P \in \mathcal{X}$ satisfying*

$$APA^* - \tilde{Z}^*P\tilde{Z} \prec 0. \tag{9}$$

The following lemma asserts that strongly ℓ_2-stable NLPV models constitute a subset of the ℓ_2-stable ones.

Lemma 7 *A strongly ℓ_2-stable NLPV model is also ℓ_2-stable; however, the converse is not true in general.*

The proof parallels the standard case and is hence omitted.

Remark 2 *We know that, under ℓ_2-stability, $\|(I - \Delta\tilde{Z}A)^{-1}\|$ is bounded for all $\Delta \in \boldsymbol{\Delta}$. But, this boundedness is* not *necessarily uniform. On the other hand, strong ℓ_2-stability guarantees that the aforementioned norm is uniformly bounded; this is clearly shown by the following norm condition which is easily derived:*

$$\|(I - \Delta\tilde{Z}A)^{-1}\| \leq \frac{\|P^{-\frac{1}{2}}\| \cdot \|P^{\frac{1}{2}}\|}{1 - \|P^{-\frac{1}{2}}\tilde{Z}AP^{\frac{1}{2}}\|} < \infty \quad \text{for all } \Delta \in \boldsymbol{\Delta},$$

where P is any solution in $\mathcal{X}$ to inequality (9).

One of the key features of strongly ℓ_2-stable NLPV models is that they can always be represented by an equivalent *balanced* realization, as we will show next. But first, we need to define the balanced realizations of an NLPV model.

Definition 23 *An NLPV system realization is* ***balanced*** *if there exists a* ***diagonal*** *operator $\Sigma \in \mathcal{X}$ satisfying*

$$A\Sigma A^* - \tilde{Z}^*\Sigma\tilde{Z} + BB^* \prec 0, \tag{10}$$

$$A^*\tilde{Z}^*\Sigma\tilde{Z}A - \Sigma + C^*C \prec 0. \tag{11}$$

Lemma 8 *An NLPV model can be equivalently represented by a balanced realization if and only if it is strongly ℓ_2-stable.*

Proof. Consider a strongly ℓ_2-stable NLPV model $(A, B, C, D; \boldsymbol{\Delta})$. This is equivalent to the existence of $P \in \mathcal{X}$ satisfying (9), which in turn is equivalent to the existence of operators $X, Y \in \mathcal{X}$ solving the generalized Lyapunov inequalities

$$AXA^* - \tilde{Z}^*X\tilde{Z} + BB^* \prec 0, \qquad A^*\tilde{Z}^*Y\tilde{Z}A - Y + C^*C \prec 0.$$

Clearly, these conditions are themselves equivalent. Now we define the operator $T \in \mathcal{T}$ by

$$T := \Sigma^{\frac{1}{2}}U^*X^{-\frac{1}{2}},$$

where unitary operator $U \in \mathcal{T}$ and positive definite diagonal operator Σ are obtained by performing a singular value decomposition on $X^{\frac{1}{2}}YX^{\frac{1}{2}}$, namely $U\Sigma^2U^* = X^{\frac{1}{2}}YX^{\frac{1}{2}}$. Then, the following holds:

$$TXT^* = (T^*)^{-1}YT^{-1} = \Sigma.$$

The equivalent realization $\left((\tilde{Z}^*T\tilde{Z})AT^{-1}, (\tilde{Z}^*T\tilde{Z})B, CT^{-1}, D; \boldsymbol{\Delta}\right)$ as a result is obviously balanced.

4 Model Reduction of Strongly $\boldsymbol{\ell_2}$-Stable NLPV Systems

This section focuses on the balanced truncation model reduction of strongly ℓ_2-stable NLPV systems. It is divided into three subsections: the first presents a precise formulation of the balanced truncation problem; the second gives upper bounds on the error induced in such a reduction process; and the last deals with eventually periodic NLPV systems and delivers guaranteed finite error bounds for the balanced truncation of such systems.

4.1 Balanced Truncation

Consider the balanced NLPV realization $(A, B, C, D; \boldsymbol{\Delta})$ with generalized diagonal gramian $\Sigma \in \mathcal{X}$ satisfying both of the generalized Lyapunov inequalities (10) and (11). Recall that Σ is of the form

$$\Sigma = \begin{bmatrix} \Sigma_0 & & & \\ & \Sigma_1 & & \\ & & \ddots & \\ & & & \Sigma_d \end{bmatrix}, \quad \text{where each} \Sigma_i = \begin{bmatrix} \Sigma_i(0) & & & \\ & \Sigma_i(1) & & \\ & & \Sigma_i(2) & \\ & & & \ddots \end{bmatrix},$$

and $\Sigma_i(k)$ is a diagonal positive definite matrix in $\mathbb{S}^{n_i(k)}$. We assume without loss of generality that, in each block $\Sigma_i(k)$, the diagonal entries are ordered with the largest first. Now given the integers $r_i(k)$ such that $0 \leq r_i(k) \leq n_i(k)$ for all $k \geq 0$, we partition each of the matrices $\Sigma_i(k)$ into two sub-blocks $\Gamma_i(k) \in \mathbb{S}^{r_i(k)}$ and $\Omega_i(k) \in \mathbb{S}^{n_i(k)-r_i(k)}$ so that

$$\Sigma_i = \left[\!\left[\begin{matrix} \Gamma_i & 0 \\ 0 & \Omega_i \end{matrix}\right]\!\right], \tag{12}$$

where Γ_i and Ω_i are block-diagonal operators. Note that, since $r_i(k)$ is allowed to be equal to zero or $n_i(k)$ at any time k, it is possible to have one of the matrices $\Omega_i(k)$ or $\Gamma_i(k)$ with zero dimension; this corresponds to the case where either zero states or all states are truncated at a particular k. Allowing for matrices with no entries, although a slight abuse of notation, will be very helpful in the manipulations of the sequel. We define the operators Γ and Ω to have a similar structure to that of Σ, namely $\Gamma = \text{diag}(\Gamma_0, \Gamma_1, \ldots, \Gamma_d)$ and $\Omega = \text{diag}(\Omega_0, \Omega_1, \ldots, \Omega_d)$. The singular values corresponding to the states and parameters that will be truncated are in Ω.

At this point, we want to partition A, B and C conformably with the partitioning of Σ. Recall from Section 3 that A, B and C have the following forms:

$$A = \begin{bmatrix} A_{ss} & A_{sp}^1 & \cdots & A_{sp}^d \\ A_{ps}^1 & A_{pp}^{11} & \cdots & A_{pp}^{1d} \\ \vdots & \vdots & \ddots & \vdots \\ A_{ps}^d & A_{pp}^{d1} & \cdots & A_{pp}^{dd} \end{bmatrix}, \quad B = \begin{bmatrix} B_s \\ B_p^1 \\ \vdots \\ B_p^d \end{bmatrix}, \quad C = \begin{bmatrix} C_s & C_p^1 & \cdots & C_p^d \end{bmatrix},$$

where each of the elements of these partitioned system operators is block-diagonal. Note further that

$$[\![A]\!]_k = \begin{bmatrix} A_{ss}(k) & A_{sp}^1(k) & \cdots & A_{sp}^d(k) \\ A_{ps}^1(k) & A_{pp}^{11}(k) & \cdots & A_{pp}^{1d}(k) \\ \vdots & \vdots & \ddots & \vdots \\ A_{ps}^d(k) & A_{pp}^{d1}(k) & \cdots & A_{pp}^{dd}(k) \end{bmatrix}, \quad [\![B]\!]_k = \begin{bmatrix} B_s(k) \\ B_p^1(k) \\ \vdots \\ B_p^d(k) \end{bmatrix},$$
$$[\![C]\!]_k = \begin{bmatrix} C_s(k) & C_p^1(k) & \cdots & C_p^d(k) \end{bmatrix}. \tag{13}$$

Let us now focus on the matrices $A_{ss}(k)$, $B_s(k)$, $C_s(k)$ and partition them in accordance with the partitionings of $\Sigma_0(k) = \mathrm{diag}(\Gamma_0(k), \Omega_0(k))$ and $\Sigma_0(k+1) = \mathrm{diag}(\Gamma_0(k+1), \Omega_0(k+1))$ so that

$$A_{ss}(k) = \begin{bmatrix} \hat{A}_{ss}(k) & A_{ss_{12}}(k) \\ A_{ss_{21}}(k) & A_{ss_{22}}(k) \end{bmatrix}, \quad B_s(k) = \begin{bmatrix} \hat{B}_s(k) \\ B_{s_2}(k) \end{bmatrix}, \quad C_s(k) = \begin{bmatrix} \hat{C}_s(k) & C_{s_2}(k) \end{bmatrix},$$

where $\hat{A}_{ss}(k) \in \mathbb{R}^{r_0(k+1)\times r_0(k)}$, $\hat{B}_s(k) \in \mathbb{R}^{r_0(k+1)\times n_w(k)}$, and $\hat{C}_s(k) \in \mathbb{R}^{n_z(k)\times r_0(k)}$. Hence, we have

$$A_{ss} = \left[\!\!\left[\begin{matrix} \hat{A}_{ss} & A_{ss_{12}} \\ A_{ss_{21}} & A_{ss_{22}} \end{matrix}\right]\!\!\right], \quad B_s = \left[\!\!\left[\begin{matrix} \hat{B}_s \\ B_{s_2} \end{matrix}\right]\!\!\right], \quad C_s = [\![\hat{C}_s \; C_{s_2}]\!],$$

where each of the elements is block-diagonal. Similarly, the other elements of the system matrices in (13) are partitioned compatibly with the partitioning of the associated $\Sigma_i(k)$ so that

$$A = \begin{bmatrix} \left[\!\!\left[\begin{matrix} \hat{A}_{ss} & A_{ss_{12}} \\ A_{ss_{21}} & A_{ss_{22}} \end{matrix}\right]\!\!\right] & \left[\!\!\left[\begin{matrix} \hat{A}^1_{sp} & A^1_{sp_{12}} \\ A^1_{sp_{21}} & A^1_{sp_{22}} \end{matrix}\right]\!\!\right] & \cdots & \left[\!\!\left[\begin{matrix} \hat{A}^d_{sp} & A^d_{sp_{12}} \\ A^d_{sp_{21}} & A^d_{sp_{22}} \end{matrix}\right]\!\!\right] \\ \left[\!\!\left[\begin{matrix} \hat{A}^1_{ps} & A^1_{ps_{12}} \\ A^1_{ps_{21}} & A^1_{ps_{22}} \end{matrix}\right]\!\!\right] & \left[\!\!\left[\begin{matrix} \hat{A}^{11}_{pp} & A^{11}_{pp_{12}} \\ A^{11}_{pp_{21}} & A^{11}_{pp_{22}} \end{matrix}\right]\!\!\right] & \cdots & \left[\!\!\left[\begin{matrix} \hat{A}^{1d}_{pp} & A^{1d}_{pp_{12}} \\ A^{1d}_{pp_{21}} & A^{1d}_{pp_{22}} \end{matrix}\right]\!\!\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[\!\!\left[\begin{matrix} \hat{A}^d_{ps} & A^d_{ps_{12}} \\ A^d_{ps_{21}} & A^d_{ps_{22}} \end{matrix}\right]\!\!\right] & \left[\!\!\left[\begin{matrix} \hat{A}^{d1}_{pp} & A^{d1}_{pp_{12}} \\ A^{d1}_{pp_{21}} & A^{d1}_{pp_{22}} \end{matrix}\right]\!\!\right] & \cdots & \left[\!\!\left[\begin{matrix} \hat{A}^{dd}_{pp} & A^{dd}_{pp_{12}} \\ A^{dd}_{pp_{21}} & A^{dd}_{pp_{22}} \end{matrix}\right]\!\!\right] \end{bmatrix}, \quad B = \begin{bmatrix} \left[\!\!\left[\begin{matrix} \hat{B}_s \\ B_{s_2} \end{matrix}\right]\!\!\right] \\ \left[\!\!\left[\begin{matrix} \hat{B}^1_p \\ B^1_{p_2} \end{matrix}\right]\!\!\right] \\ \vdots \\ \left[\!\!\left[\begin{matrix} \hat{B}^d_p \\ B^d_{p_2} \end{matrix}\right]\!\!\right] \end{bmatrix},$$

$$C = \begin{bmatrix} [\![\hat{C}_s \; C_{s_2}]\!] & [\![\hat{C}^1_p \; C^1_{p_2}]\!] & \cdots & [\![\hat{C}^d_p \; C^d_{p_2}]\!] \end{bmatrix}.$$

Then a state space realization for the balanced truncation $G_{\delta,r}$ of the system G_δ is $(A_r, B_r, C_r, D_r; \boldsymbol{\Delta}_r)$ where

$$\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} = \begin{bmatrix} \hat{A}_{ss} & \hat{A}^1_{sp} & \cdots & \hat{A}^d_{sp} & \hat{B}_s \\ \hat{A}^1_{ps} & \hat{A}^{11}_{pp} & \cdots & \hat{A}^{1d}_{pp} & \hat{B}^1_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{A}^d_{ps} & \hat{A}^{d1}_{pp} & \cdots & \hat{A}^{dd}_{pp} & \hat{B}^d_p \\ \hat{C}_s & \hat{C}^1_p & \cdots & \hat{C}^d_p & D \end{bmatrix},$$

and $\Delta_r = \mathrm{diag}(I^{r_0}_{\ell_2}, \hat{\Delta}_1, \ldots, \hat{\Delta}_d)$, with $\hat{\Delta}_i = \mathrm{diag}(\delta_i(0)I_{r_i(0)}, \delta_i(1)I_{r_i(1)}, \ldots)$. Notice that Δ_r is constructed from the same parameters δ_i as those in Δ.

Lemma 9 *Suppose* $(A, B, C, D; \boldsymbol{\Delta})$ *is a balanced realization of* G_δ*. Then the corresponding balanced truncation* $G_{\delta,r}$ *is also strongly* ℓ_2*-stable and balanced.*

Proof. To start, there exists a unique permutation Q such that $Q^*\Sigma Q = \mathrm{diag}(\Gamma, \Omega)$ formally; then we have

$$Q^*\tilde{Z}AQ = \begin{bmatrix} \tilde{Z} & \\ & \tilde{Z} \end{bmatrix} \begin{bmatrix} A_r & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \tilde{Z}_2\bar{A}, \quad Q^*\tilde{Z}B = \begin{bmatrix} \tilde{Z} & \\ & \tilde{Z} \end{bmatrix} \begin{bmatrix} B_r \\ \bar{B}_2 \end{bmatrix} = \tilde{Z}_2\bar{B},$$
$$CQ = \begin{bmatrix} C_r & \bar{C}_2 \end{bmatrix} = \bar{C},$$

and $\bar{\Delta} = Q^*\Delta Q = \text{diag}(\Delta_r, \bar{\Delta}_2)$, where $(A_r, B_r, C_r, D_r; \Delta_r)$ is a realization of the truncation $G_{\delta,r}$, and the rest of the operators are defined in the obvious way.

As the generalized gramian Σ satisfies both of inequalities (10) and (11), then focusing on (10), and with the aforesaid permutation in mind, the following ensues:

$$\begin{bmatrix} A_r & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \Gamma & \\ & \Omega \end{bmatrix} \begin{bmatrix} A_r^* & \bar{A}_{21}^* \\ \bar{A}_{12}^* & \bar{A}_{22}^* \end{bmatrix} - \begin{bmatrix} \tilde{Z}^*\Gamma\tilde{Z} & \\ & \tilde{Z}^*\Omega\tilde{Z} \end{bmatrix} + \begin{bmatrix} B_r \\ \bar{B}_2 \end{bmatrix} \begin{bmatrix} B_r^* & \bar{B}_2^* \end{bmatrix} \prec 0.$$

This clearly gives

$$A_r\Gamma A_r^* - \tilde{Z}^*\Gamma\tilde{Z} + B_rB_r^* \prec 0.$$

Similarly, starting with (11), we can show that

$$A_r^*\tilde{Z}^*\Gamma\tilde{Z}A_r - \Gamma + C_r^*C_r \prec 0.$$

Thus directly from the definitions of strong stability and a balanced system we have the desired conclusion.

4.2 Error Bounds

This subsection gives upper bounds on the error induced in the balanced truncation model reduction process. We start with the following result.

Lemma 10 *An NLPV model G_δ is strongly ℓ_2-stable and satisfies the condition $\|G_\delta\| < 1$, for all $\Delta \in \boldsymbol{\Delta}$, if there exists a positive definite operator V in the commutant of $\boldsymbol{\Delta}$ such that*

$$\begin{bmatrix} -V & 0 & A^* & C^* \\ 0 & -I & B^* & D^* \\ A & B & -\tilde{Z}^*V^{-1}\tilde{Z} & 0 \\ C & D & 0 & -I \end{bmatrix} \prec 0. \tag{14}$$

This is a generalization of the sufficiency part of the Kalman-Yakubovich-Popov (KYP) Lemma. Its proof is routine and so we do not include it here. Note that the above inequality is necessary and sufficient in the purely time-varying case as proved in [19]; however, in our case, it is in general only sufficient. We will find the following notation convenient:

$$G_\delta = \Delta \star G = C(I - \Delta\tilde{Z}A)^{-1}\Delta\tilde{Z}B + D,$$
$$\text{where} \quad G = \begin{bmatrix} \tilde{Z}A & \tilde{Z}B \\ C & D \end{bmatrix} \text{ and } \Delta \in \boldsymbol{\Delta}. \tag{15}$$

Theorem 24 *Suppose that $(A, B, C, D; \boldsymbol{\Delta})$ is a balanced realization for the NLPV system G_δ, and that the diagonal generalized gramian $\Sigma \in \mathcal{X}$, satisfying both of inequalities (10) and (11), is partitioned as in (12). If $\Omega_i = I_{\ell_2}$ for all $i = 0, 1, \ldots, d$, then, for all $\Delta \in \boldsymbol{\Delta}$, the balanced truncation $G_{\delta,r}$ of G_δ satisfies the following inequality:*

$$\|G_\delta - G_{\delta,r}\| < 2.$$

Proof. As G_δ and $G_{\delta,r}$ are both strongly ℓ_2-stable, then so is $\frac{1}{2}(G_\delta - G_{\delta,r})$. One realization of the system $\frac{1}{2}(G_\delta - G_{\delta,r})$ is given in linear fractional form by

$$\frac{1}{2}(G_\delta - G_{\delta,r}) = \begin{bmatrix} \Delta_r & \\ & \bar{\Delta} \end{bmatrix} \star \begin{bmatrix} \tilde{Z}A_r & 0 & \frac{1}{\sqrt{2}}\tilde{Z}B_r \\ 0 & \tilde{Z}_2\bar{A} & \frac{1}{\sqrt{2}}\tilde{Z}_2\bar{B} \\ -\frac{1}{\sqrt{2}}C_r & \frac{1}{\sqrt{2}}\bar{C} & 0 \end{bmatrix},$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{\Delta}$ are as defined in the proof of Lemma 9, and $\tilde{Z}_m = \operatorname{diag}(\{J_i\}_{i=1}^m)$ where $J_i = \tilde{Z}$ for all i. In the sequel, we will construct a positive definite operator V that commutes with $\operatorname{diag}(\Delta_r, \bar{\Delta})$ and satisfies inequality (14) for this $\frac{1}{2}(G_\delta - G_{\delta,r})$ realization. Then, invoking Lemma 10 completes the proof.

Given that the diagonal operator $\Sigma \in \mathcal{X}$ satisfies inequalities (10) and (11), then direct applications of the Schur complement formula, along with some permutations, guarantee the validity of the following condition:

$$\begin{bmatrix} -R_1 & K^* \\ K & -\bar{\mathcal{Z}}_a^* R_2 \bar{\mathcal{Z}}_a \end{bmatrix} \prec 0,$$

where $\bar{\mathcal{Z}}_a = \operatorname{diag}(\tilde{Z}_2, I, \tilde{Z}_2)$,

$$R_i = \begin{bmatrix} \Gamma^{-1} & 0 & 0 & 0 & 0 \\ 0 & \Omega^{-1} & 0 & 0 & 0 \\ 0 & 0 & I_{\ell_2}^{q_i} & 0 & 0 \\ 0 & 0 & 0 & \Gamma & 0 \\ 0 & 0 & 0 & 0 & \Omega \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & 0 & A_r & \bar{A}_{12} \\ 0 & 0 & 0 & \bar{A}_{21} & \bar{A}_{22} \\ 0 & 0 & 0 & C_r & \bar{C}_2 \\ A_r & \bar{A}_{12} & B_r & 0 & 0 \\ \bar{A}_{21} & \bar{A}_{22} & \bar{B}_2 & 0 & 0 \end{bmatrix},$$

and $q_1 = n_w$, $q_2 = n_z$. Define the invertible operators L and S by

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} -I & 0 & 0 & I & 0 \\ I & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & I \\ 0 & 0 & \sqrt{2}I & 0 & 0 \\ 0 & -I & 0 & 0 & I \end{bmatrix}, \quad S = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & I \\ 0 & 0 & 0 & \sqrt{2}I & 0 \\ -I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -I \end{bmatrix}.$$

Multiplying the preceding condition on the left by $\operatorname{diag}(S^*, L)$ and on the right by $\operatorname{diag}(S, L^*)$ gives the following equivalent inequality:

$$\begin{bmatrix} -S^* R_1 S & S^* K^* L^* \\ LKS & -\bar{\mathcal{Z}}_b^* L R_2 L^* \bar{\mathcal{Z}}_b \end{bmatrix} \prec 0, \tag{16}$$

where $\bar{\mathcal{Z}}_b = \operatorname{diag}(\tilde{Z}_3, I, \tilde{Z})$. Performing the multiplications in this inequality leads to

$$S^* R_1 S = \begin{bmatrix} \frac{1}{2}(\Gamma^{-1}+\Gamma) & \frac{1}{2}(\Gamma^{-1}-\Gamma) & 0 & 0 & 0 \\ \frac{1}{2}(\Gamma^{-1}-\Gamma) & \frac{1}{2}(\Gamma^{-1}+\Gamma) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\Omega^{-1}+\Omega) & 0 & \frac{1}{2}(\Omega^{-1}-\Omega) \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & \frac{1}{2}(\Omega^{-1}-\Omega) & 0 & \frac{1}{2}(\Omega^{-1}+\Omega) \end{bmatrix},$$

$$L R_2 L^* = \begin{bmatrix} \frac{1}{2}(\Gamma^{-1}+\Gamma) & \frac{1}{2}(\Gamma-\Gamma^{-1}) & 0 & 0 & 0 \\ \frac{1}{2}(\Gamma-\Gamma^{-1}) & \frac{1}{2}(\Gamma^{-1}+\Gamma) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\Omega^{-1}+\Omega) & 0 & \frac{1}{2}(\Omega-\Omega^{-1}) \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & \frac{1}{2}(\Omega-\Omega^{-1}) & 0 & \frac{1}{2}(\Omega^{-1}+\Omega) \end{bmatrix},$$

$$LKS = \begin{bmatrix} A_r & 0 & 0 & \frac{1}{\sqrt{2}}B_r & \bar{A}_{12} \\ 0 & A_r & \bar{A}_{12} & \frac{1}{\sqrt{2}}B_r & 0 \\ 0 & \bar{A}_{21} & \bar{A}_{22} & \frac{1}{\sqrt{2}}\bar{B}_2 & 0 \\ -\frac{1}{\sqrt{2}}C_r & \frac{1}{\sqrt{2}}C_r & \frac{1}{\sqrt{2}}\bar{C}_2 & 0 & -\frac{1}{\sqrt{2}}\bar{C}_2 \\ \bar{A}_{21} & 0 & 0 & \frac{1}{\sqrt{2}}\bar{B}_2 & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} M & N_{12} \\ N_{21} & \bar{A}_{22} \end{bmatrix}.$$

Note that, in the preceding expressions, some of the operators might contain at certain time-instants matrices of zero dimensions. In such scenarios, the rows and columns of which the said matrices are elements would not be present, and the corresponding operator inequalities remain valid.

Define the operator V as

$$V = \begin{bmatrix} \frac{1}{2}(\Gamma^{-1}+\Gamma) & \frac{1}{2}(\Gamma^{-1}-\Gamma) & 0 \\ \frac{1}{2}(\Gamma^{-1}-\Gamma) & \frac{1}{2}(\Gamma^{-1}+\Gamma) & 0 \\ 0 & 0 & \frac{1}{2}(\Omega^{-1}+\Omega) \end{bmatrix}.$$

Note that, since $S^* R_1 S \succ 0$, then $V \succ 0$. Also, V clearly commutes with the operator $\operatorname{diag}(\Delta_r, \bar{\Delta})$. Recall that, by assumption, $\Omega = I$; hence, $\frac{1}{2}(\Omega^{-1} + \Omega) = I$ and $\frac{1}{2}(\Omega^{-1} - \Omega) = 0$. With this in mind, it is not difficult to see that inequality (16) implies that

$$\begin{bmatrix} -\begin{bmatrix} V & \\ & I_{\ell_2}^{n_w} \end{bmatrix} & M^* \\ M & -\begin{bmatrix} \tilde{Z}_3^* V^{-1} \tilde{Z}_3 & \\ & I_{\ell_2}^{n_z} \end{bmatrix} \end{bmatrix} \prec 0.$$

Then, invoking Lemma 10, we get $\|\frac{1}{2}\left(G_\delta - G_{\delta,r}\right)\| < 1$.

Theorem 25 *Given a balanced NLPV model G_δ, then, for all $\Delta \in \boldsymbol{\Delta}$, its balanced truncation $G_{\delta,r}$ satisfies the error bound*

$$\|G_\delta - G_{\delta,r}\| < 2\sum_{i=0}^{d}\sum_{j}\omega_{i,j},$$

where $\omega_{i,j}$ are the distinct diagonal entries of the block-diagonal operator Ω_i.

The proof follows from scaling, Lemma 9 and repeated application of the previous theorem. Note that this error bound might involve an infinite summation which in general may not converge to a finite number. In the following, we improve on this result and derive tighter bounds. We will first consider balanced systems where the singular values corresponding to the states and parameters to be truncated are monotonic in time.

Before doing this it will be convenient to establish the following terminology.

Definition 26 *Given a scalar sequence α_k defined on a subset $\mathcal{W}$ of the non-negative integers, we define the following* ***hold rule*** *which extends the domain of α_k to all $k \geq 0$: let $k_{\min} = \min\{k \geq 0 : k \in \mathcal{W}\}$ and then set*

$$\alpha_k = \begin{cases} \alpha_{k_{\min}}, & \text{if } 0 \leq k \leq k_{\min}; \\ \alpha_q, \ \text{where } q := \max\{q \leq k : q \in \mathcal{W}\}, & \text{if } k_{\min} < k. \end{cases}$$

We now have the following result.

Theorem 27 (monotonic case) *Suppose that $(A, B, C, D; \boldsymbol{\Delta})$ is a balanced realization for the NLPV system G_δ, and that the diagonal generalized gramian $\Sigma \in \mathcal{X}$, satisfying both of inequalities (10) and (11), is partitioned as in (12). Let $s_i(k) = n_i(k) - r_i(k)$ and define the set $\mathcal{F}_i$, for $i = 0, 1, \ldots, d$, by*

$$\mathcal{F}_i = \{k \geq 0 : s_i(k) > 0\}.$$

Also suppose that for each $i = 0, 1, \ldots, d$ the scalar sequence $\omega_{i,k}$ satisfies $\Omega_i(k) = \omega_{i,k} I_{s_i(k)}$ for all $k \in \mathcal{F}_i$.
If for each $i = 0, 1, \ldots, d$ the sequence $\omega_{i,k}$ is monotonic on $\mathcal{F}_k$, then for all $\Delta \in \boldsymbol{\Delta}$ the balanced truncation $G_{\delta,r}$ of G_δ satisfies the following inequality:

$$\|G_\delta - G_{\delta,r}\| < 2\sum_{i=0}^{d} \sup_{k \in \mathcal{F}_i} \omega_{i,k}.$$

Proof. It is sufficient to prove the theorem for the case where only one parameter or state block is being truncated (i.e., $s_i = 0$ for all i except for one, say $j \in \{0, 1, 2, \ldots, d\}$), since the general case then follows simply by the standard use of the telescoping series and triangle inequality. Also, we assume without loss of generality that $\omega_{j,k} \leq 1$ for all $k \in \mathcal{F}_j$; this can always be achieved by scaling inequalities (10) and (11).

To begin, we extend the domain of definition of $\omega_{j,k}$ to all $k \geq 0$ using the hold rule defined in Definition 26; note that the extended sequence is still monotonic. We now split the remainder of our proof into two separate cases, one where this sequence is nondecreasing and the other where it is nonincreasing.

Case $\omega_{j,k}$ nondecreasing:

In this case, we have $\omega_{j,k} \leq \omega_{j,k+1}$ for all $k \geq 0$. We define the state space transformation $T \in \mathcal{T}$ as

$$\llbracket T \rrbracket_k = (\omega_{j,k})^{-\frac{1}{2}} I. \tag{17}$$

Note that, since $\Sigma \succ 0$, then T is indeed bounded. This gives the following balanced realization for G_δ:

$$(\bar{A}, \bar{B}, \bar{C}, D; \boldsymbol{\Delta}) := \left(\left(\tilde{Z}^* T \tilde{Z}\right) A T^{-1}, \left(\tilde{Z}^* T \tilde{Z}\right) B, C T^{-1}, D; \boldsymbol{\Delta}\right). \tag{18}$$

For convenient reference, we will use $\bar{G}_\delta$ to refer to the system G_δ when the realization in use is (18).

Our goal now is to show that this new realization is balanced. To this end, given the state transformation T, we use (10) and (11) to arrive at

$$\begin{aligned} \bar{A}\bar{\Sigma}\bar{A}^* - \tilde{Z}^*\bar{\Sigma}\tilde{Z} + \bar{B}\bar{B}^* \prec 0, \\ \bar{A}^*\tilde{Z}^*\left((T^*)^{-1}\Sigma T^{-1}\right)\tilde{Z}\bar{A} - (T^*)^{-1}\Sigma T^{-1} + \bar{C}^*\bar{C} \prec 0, \end{aligned} \tag{19}$$

where $\bar{\Sigma} = T\Sigma T^*$. Because of the special structure of T and the fact that $\omega_{j,k} \leq \omega_{j,k+1} \leq 1$, it is not difficult to see that

$$\bar{C}^*\bar{C} = (T^*)^{-1}C^*CT^{-1} \preceq T^*C^*CT,$$

$$\bar{A}^*\tilde{Z}^*\bar{\Sigma}\tilde{Z}\bar{A} = \bar{A}^*\tilde{Z}^*\left(T\Sigma T^*\right)\tilde{Z}\bar{A} \preceq (T^*)^2\bar{A}^*\tilde{Z}^*\left((T^*)^{-1}\Sigma T^{-1}\right)\tilde{Z}\bar{A}T^2.$$

Then, pre- and post-multiplying inequality (19) by $(T^*)^2$ and T^2 respectively and then using the above inequalities give

$$\bar{A}^*\tilde{Z}^*\bar{\Sigma}\tilde{Z}\bar{A} - \bar{\Sigma} + \bar{C}^*\bar{C} \prec 0.$$

Hence, $\bar{\Sigma}$ is a diagonal gramian satisfying the generalized Lyapunov inequalities for the system realization $\bar{G}_\delta$. Notice that, by the definition of T, we have

$$\begin{aligned} \bar{\Sigma}_j(k) = T_j(k)\Sigma_j(k)T_j^*(k) &= (\omega_{j,k})^{-1}\begin{bmatrix} \Gamma_j(k) & \\ & \Omega_j(k) \end{bmatrix} \\ &= \begin{bmatrix} (\omega_{j,k})^{-1}\Gamma_j(k) & \\ & I \end{bmatrix} = \begin{bmatrix} \bar{\Gamma}_j(k) & \\ & \bar{\Omega}_j(k) \end{bmatrix}. \end{aligned}$$

Thus, $\bar{\Omega}_j = I_{\ell_2}$, and so, by invoking Theorem 24, we deduce that the balanced truncation $\bar{G}_{\delta,r}$ of the system $\bar{G}_\delta$ satisfies the norm condition

$$\|\bar{G}_\delta - \bar{G}_{\delta,r}\| < 2$$

for all $\Delta \in \boldsymbol{\Delta}$. Now, it is not difficult to see that, because of the special structure of T, the error system realizations $G_\delta - G_{\delta,r}$ and $\bar{G}_\delta - \bar{G}_{\delta,r}$ are in fact equivalent, and as a result, we have

$$\|G_\delta - G_{\delta,r}\| = \|\bar{G}_\delta - \bar{G}_{\delta,r}\| < 2.$$

Case $\omega_{j,k}$ nonincreasing:

A similar argument applies where here the state transformation $T \in \mathcal{T}$ is defined as $[\![T]\!]_k = (\omega_{j,k})^{\frac{1}{2}} I$.

We now consider the more general case, where singular values need not be monotonic in time. But first, we require the following definition from [18].

Definition 28 *Given a vector $v = (v_1, v_2, \ldots, v_s)$ for some integer $s \geq 1$, suppose that v_1 cannot be considered as a local maximum and v_s cannot be considered as a local minimum. Then vector v has m local maxima $v_{\max,i}$ and m local minima $v_{\min,i}$ for some integer $m \geq 0$, and the max-min ratio of v, denoted $\mathcal{S}_v$, is defined as*

$$\mathcal{S}_v = v_1 \prod_{i=1}^{m} \frac{v_{\max,i}}{v_{\min,i}}, \qquad m > 0$$

$$\mathcal{S}_v = v_1, \qquad m = 0.$$

Theorem 29 (nonmonotonic case) *Given a balanced realization $(A, B, C, D; \boldsymbol{\Delta})$ for the NLPV system G_δ, suppose that a diagonal operator $\Sigma \in \mathcal{X}$ satisfies both of inequalities (10) and (11) and is partitioned as in (12), where, for all $i = 0, 1, \ldots, d$ and $k \in \mathcal{F}_i = \{k \geq 0 : s_i(k) > 0\}$ we have $\Omega_i(k) = \omega_{i,k} I_{s_i(k)}$, with $s_i(k) := n_i(k) - r_i(k)$. Define the vector $\hat{\omega}_i$ to consist of the elements $\omega_{i,k}$ for $k \in \mathcal{F}_i$.*
If for each $i = 1, \ldots, d$ we have $\dim(\hat{\omega}_i) < \infty$, then for all $\Delta \in \boldsymbol{\Delta}$ the balanced truncation $G_{\delta,r}$ of G_δ satisfies the following inequality:

$$\|G_\delta - G_{\delta,r}\| < 2 \sum_{i=0}^{d} \mathcal{S}_{\hat{\omega}_i}.$$

The proof below uses the same idea as that of Theorem 27. Basically, we need to define some state space transformation that results in a balanced realization for the system G_δ where the diagonal gramian $\bar{\Sigma}$ solving the generalized Lyapunov inequalities for this realization is such that $\bar{\Omega}_j = I_{\ell_2}$. Then, invoking Theorem 24 completes the proof. The choice of this state space transformation used is inspired by that of the monotonic case.

Proof. As with the proof of Theorem 27 it is sufficient to prove the result for the case where the only Ω_i that has non-zero dimension is Ω_j for some fixed j in $\{0, 1, \ldots, d\}$; without loss of generality we assume that $\omega_{j,k} \leq 1$ for all k.

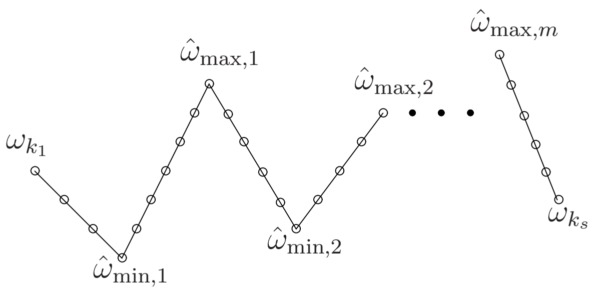
To keep the notation simple, we suppress the subscript j in ω_j and $\hat{\omega}_j$. The vector $\hat{\omega}$ is of the form

$$\hat{\omega} = (\omega_{k_1}, \omega_{k_2}, \ldots, \hat{\omega}_{\min,1}, \ldots, \hat{\omega}_{\max,1}, \ldots \ \ldots, \hat{\omega}_{\max,m}, \ldots, \omega_{k_s}),$$

corresponding to values of the sequence ω_k evaluated at the ordered time points

$$(k_1, k_2, \ldots, k_{\min,1}, \ldots, k_{\max,1}, \ldots, k_{\max,m}, \ldots, k_s).$$

The denoted local minima and maxima of the vector $\hat{\omega}$ are as defined in Definition 28. We now use the hold rule of Definition 26 to extend the sequence ω_k to all $k \geq 0$; the maxima and minima of ω_k are illustrated below.



We define the state space transformation $T \in \mathcal{T}$ as

$$[\![T]\!]_k = \begin{cases} \omega_{k_1}^{\frac{1}{2}} I, & \text{for } k = 0, 1, \ldots, k_1 - 1, \\ \omega_k^{\frac{1}{2}} I, & \text{for } k = k_1, k_1 + 1, \ldots, k_{\min,1}, \\ \omega_{\min,1}\omega_k^{-\frac{1}{2}} I, & \text{for } k = k_{\min,1} + 1, \ldots, k_{\max,1}, \\ \omega_{\min,1}\omega_{\max,1}^{-1}\omega_k^{\frac{1}{2}} I, & \text{for } k = k_{\max,1} + 1, \ldots, k_{\min,2}, \\ \quad\vdots & \qquad\vdots \\ \rho\omega_k^{\frac{1}{2}} I, & \text{for } k = k_{\max,m} + 1, \ldots, k_s, \\ \rho\omega_{k_s}^{\frac{1}{2}} I, & \text{for } k = k_s + 1, k_s + 2, \ldots, \end{cases}$$

where $\rho = \prod_{i=1}^{m} \omega_{\min,i}\omega_{\max,i}^{-1}$. Also, define $P, Q \in \mathcal{T}$ such that $[\![P]\!]_k = \omega_k^{\frac{1}{2}} I$ and $Q = TP^{-1}$. It is not difficult to see that the constituent scalars of operator T define a nonincreasing sequence, and so do those of operator Q and those of operator QP^2. Then, given the equivalent realization $\left(\bar{A}, \bar{B}, \bar{C}, D; \boldsymbol{\Delta}\right) = \left(\left(\tilde{Z}^* T \tilde{Z}\right) AT^{-1}, \left(\tilde{Z}^* T \tilde{Z}\right) B, CT^{-1}, D; \boldsymbol{\Delta}\right)$ of the system G_δ, which we denote for ease of reference by $\bar{G}_\delta$, and because of the special structure of T and the assumption that $\omega_{j,k} \leq 1$, the following ensues:

$$\bar{A}\bar{\Sigma}\bar{A}^* - \tilde{Z}^*\bar{\Sigma}\tilde{Z} + P^{-2}Q^{-1}\bar{B}\bar{B}^*(Q^*)^{-1}(P^*)^{-2} \prec 0,$$
$$\bar{A}^*\tilde{Z}^*\bar{\Sigma}\tilde{Z}\bar{A} - \bar{\Sigma} + Q^*\bar{C}^*\bar{C}Q \prec 0,$$

where $\bar{\Sigma} = (P^*)^{-1}\Sigma P^{-1}$. Notice that $P^{-2}Q^{-1} \succeq \omega_{k_1}^{-1} I$ and $Q \succeq \rho I$. Thus, the diagonal operator $\bar{\Sigma}$ satisfies the generalized Lyapunov inequalities (10) and (11) for the realization $(\bar{A}, \omega_{k_1}^{-1}\bar{B}, \rho\bar{C}, D; \Delta)$. As $\bar{\Omega}_j = I_{\ell_2}$, then, invoking Theorem 24, we get

$$\omega_{k_1}^{-1}\rho\|\bar{G}_\delta - \bar{G}_{\delta,r}\| < 2.$$

Finally, the special structure of operator T and the fact that $\mathcal{S}_\omega = \left(\omega_{k_1}^{-1}\rho\right)^{-1}$ lead to

$$\|G_\delta - G_{\delta,r}\| = \|\bar{G}_\delta - \bar{G}_{\delta,r}\| < 2\mathcal{S}_\omega.$$

We remark that Theorem 25 generalizes the LTV result in [14] to the NLPV framework. Also, Theorems 27 and 29 are mainly generalizations of their LTV counterparts in [18], with the important exception that the truncations in the theorems need not be restricted connected intervals. To illustrate how to apply these results, we consider the following hypothetical example. Suppose we are to truncate the states corresponding to the sequence $\Omega_0(k) = \omega_{0,k} I_{s_0(k)}$ for $k \in [1, 9]$, where

$$\{\omega_{0,k}\}_{k=1}^{9} = \{1, 0.75, 2, 1.25, 3, 1.75, 4, 2.25, 5\}.$$

Then the corresponding error bound obtained from Theorem 25 is

$$2 \times (1 + 0.75 + 2 + 1.25 + 3 + 1.75 + 4 + 2.25 + 5) = 42.$$

This is exactly the same bound that the main result of [14] would give assuming a standard LTV system. If we are to apply Theorem 29 to truncate the states in one step, then we obtain the error bound

$$2 \times 1 \times \frac{2}{0.75} \times \frac{3}{1.25} \times \frac{4}{1.75} \times \frac{5}{2.24} \approx 65.$$

This bound is quite conservative and can be significantly improved if we truncate the states in three steps and accordingly divide the sequence $\omega_{0,k}$ into the following: $\{1, 0.75, 2, 1.25\}$, $\{3, 1.75, 4, 2.25\}$, and $\{5\}$. Then, applying Theorem 29 recursively, we obtain the improved error bound

$$2 \times \left(1 \times \frac{2}{0.75} + 3 \times \frac{4}{1.75} + 5\right) \approx 29.$$

This can also be obtained from the results of [18] if the system in question is a standard LTV system. But, in our case, we can actually further improve on the last bound by dividing the sequence $\omega_{0,k}$ into the two monotonic sequences $\{1, 2, 3, 4, 5\}$ and $\{0.75, 1.25, 1.75, 2.25\}$ and then applying Theorem 27 twice to get the error bound $2 \times (5 + 2.25) = 14.5$.

4.3 Eventually Periodic LPV Systems

This subsection focuses on the balanced truncation of *eventually periodic* LPV systems. These systems are aperiodic for an initial amount of time, and then become periodic afterwards. One scenario in which they originate is when parametrizing nonlinear systems about eventually periodic trajectories. Such trajectories can be arbitrary for a finite amount of time, but then settle down into a periodic orbit; a special case of this occurs when a system transitions between two operating points. In addition to that, eventually periodic systems naturally arise when considering problems involving plants with uncertain initial states. Note that both finite horizon and periodic systems are subclasses of eventually periodic systems. We refer the reader to [6–8] for some useful results on eventually periodic models. We now give a precise definition of an eventually periodic operator.

Definition 30 *A block-diagonal mapping P on ℓ_2 is (h,q)-*eventually periodic *if, for some integers $h \geq 0$, $q \geq 1$, we have*

$$Z^q((Z^*)^h P Z^h) = ((Z^*)^h P Z^h) Z^q,$$

that is P is q-periodic after an initial transient behavior up to time h. Moreover, a partitioned operator, whose elements are block-diagonal, is (h,q)-eventually periodic if each of its block-diagonal elements is (h,q)-eventually periodic.

Theorem 31 *Suppose that state space operators A, B, and C are (h,q)-eventually periodic. Then solutions $X, Y \in \mathcal{X}$ satisfying Lyapunov inequalities (10) and (11) exist if and only if (h,q)-eventually periodic solutions $X_{eper}, Y_{eper} \in \mathcal{X}$ exist.*

The outline of the proof is as follows: first, employ a similar averaging technique to that used in [2] to show that the periodic part of any of the generalized Lyapunov inequalities admits a q-periodic solution if feasible, then, having established that, the above result follows from scaling.

Thus, if the system is strongly ℓ_2-stable and (h,q)-eventually periodic, then we can construct an (h,q)-eventually periodic balanced realization with an (h,q)-eventually periodic diagonal gramian $\Sigma \in \mathcal{X}$ satisfying Lyapunov inequalities (10) and (11).

Theorem 32 *Suppose that system G_δ is an (h,q)-eventually periodic system with a balanced realization $(A, B, C, D; \boldsymbol{\Delta})$. Then the following hold:*

(i) There exists an (h,q)-eventually periodic diagonal operator $\Sigma \in \mathcal{X}$, partitioned as in (12), satisfying both of the generalized Lyapunov inequalities (10) and (11);

(ii) The balanced truncation $G_{\delta,r}$ of G_δ is balanced and satisfies the finite error bound

$$\|G - G_r\| < E_{fh} + 2 \sum_{i=0}^{d} \sum_{j} \omega_{i,j} < \infty,$$

where the scalar parameters $\omega_{i,j}$ *are the* distinct *diagonal entries of the matrix* $\mathrm{diag}(\Omega_i(h), \ldots, \Omega_i(h+q-1))$*, and* E_{fh} *is the finite upper bound on the error induced in the balanced truncation of the finite horizon part of* G_δ *and is derived by applying Theorem 29.*

5 Conclusions

In this paper we have introduced balanced truncation model reduction for NLPV systems, and derived explicit error bounds for this procedure. Even when restricted to purely time-varying systems the results obtained provide the least conservative bounds currently available in the literature. Although there has been considerable recent achievement in the literature on model reducing nonstationary systems, which are all directly motivated by the original LTI results in [3, 11], the authors conjecture that significantly better bounds may be obtainable.

References

1. C.L. Beck, J.C. Doyle and K. Glover, "Model reduction of multidimensional and uncertain systems," *IEEE Transactions on Automatic Control*, 41:1466–1477, 1996.
2. G.E. Dullerud and S.G. Lall, "A new approach to analysis and synthesis of time-varying systems," *IEEE Transactions on Automatic Control*, 44:1486–1497, 1999.
3. D. Enns, "Model reduction with balanced realizations: An error bound and a frequency weighted generalization", *Proc. IEEE Conference on Decision and Control*, 1984.
4. M. Farhood, C.L. Beck and G.E. Dullerud, "Model reduction of nonstationary LPV systems," *Proc. IEEE Conference on Decision and Control*, 2002.
5. M. Farhood and G.E. Dullerud, "Control of nonstationary LPV systems," submitted to Automatica.
6. M. Farhood and G.E. Dullerud, "LMI tools for eventually periodic systems," *Sys. and Cont. Letters*, 47:417–432, 2002.
7. M. Farhood and G.E. Dullerud, "On the ℓ_2-induced control for eventually periodic systems," *Proceedings of the American Control Conference*, 2004.
8. M. Farhood and G.E. Dullerud, "Duality and eventually periodic systems," to appear in International Journal of Robust and Nonlinear Control.
9. M. Farhood, C.L. Beck, and G.E. Dullerud, "On the model reduction of nonstationary LPV systems," Proceedings of the American Control Conference, 2003.
10. P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H_∞ control," *International Journal of Robust and Nonlinear Control*, 4:421–448, 1991.

11. K. Glover, "All optimal hankel-norm approximations of linear multivariable systems and their L_∞-error bounds," *International Journal of Control*, 39:1115–1193, 1984.
12. D. Hinrichsen and A.J. Pritchard, "An improved error estimate for reduced-order models of discrete-time systems," *IEEE Transactions on Automatic Control*, 35:317–320, 1990.
13. S. Lall and C. Beck, "Error bounds for balanced model reduction of linear time-varying systems," submitted to IEEE Transactions on Automatic Control.
14. S. Lall and C. Beck, "Model reduction of complex systems in the linear-fractional framework," *Proc. IEEE Conference on Decision and Control*, 1999.
15. W.M. Lu, K. Zhou, and J.C. Doyle, "Stabilization of uncertain linear systems: an LFT approach," *IEEE Transactions on Automatic Control*, 41:50–65, 1996.
16. B.C. Moore, "Principal component analysis in linear systems: controllability, observability, and model reduction, " *IEEE Transactions on Automatic Control*, 26:17–34, 1981.
17. A. Packard, "Gain scheduling via linear fractional transformations," *Systems and Control Letters*, 22:79–92, 1994.
18. H. Sandberg and A. Rantzer, "Balanced truncation of linear time-varying systems," *IEEE Transactions on Automatic Control*, 49:217–229, 2004.
19. V.A. Yacubovich, "A frequency theorem for the case in which the state and control spaces are Hilbert spaces with an application to some problems of synthesis of optimal controls," *Sibirskii Mat. Zh.*, 15:639–668, 1975, (English translation in Siberian Mathematics Journal).